Additive Deformations of Hopf Algebras

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Abstract

Additive deformations of bialgebras in the sense of Wirth are deformations of the multiplication map of the bialgebra fulfilling a compatibility condition with the coalgebra structure and a continuity condition. Two problems concerning additive deformations are considered.

With a deformation theory a cohomology theory should be developed. Here a variant of the Hochschild cohomology is used. The main result in the first part of this paper is the characterization of the trivial deformations, i.e. deformations generated by a coboundary.

When one starts with a Hopf algebra, one would expect the deformed multiplications to have some analogue to the antipode, which we call deformed antipodes. We prove, that deformed antipodes always exist, explore their properties, give a formula to calculate them given the deformation and the antipode of the original Hopf algebra and show in the cocommutative case, that each deformation splits into a trivial part and into a part with constant antipodes.

1 Introduction

Deformations of algebras are closely related to cohomology as Gerstenhaber showed in his papers [4] and [5]. Suppose that $\mathcal{A}$ is an algebra and $(\mu_t)_{t\geq 0}$ a family of associative multiplications on $\mathcal{A}$, which can in any sense be written in the form

$$\mu_t(a\otimes b)=\mu(a\otimes b)+tF(a\otimes b)+\mathcal{O}(t^2),$$

where $\mu_0=\mu$ is the original multiplication of the algebra. Writing down the associativity condition for μ_t and comparing the terms of first order yields that

$$\mu(F(a\otimes b)\otimes c)+F(\mu(a\otimes b)\otimes c)=\mu(a\otimes F(b\otimes c))+F(a\otimes\mu(b\otimes c))$$

and after rearranging

$$aF(b\otimes c)-F(ab\otimes c)+F(a\otimes bc)-F(a\otimes b)c=0,$$

so the infinitesimal deformation F is a cocycle in the Hochschild cohomology associated with the $\mathcal{A}$-Bimodule structure on $\mathcal{A}$ given by multiplication.

Additive deformations were first introduced by Wirth in [10] as a generalization of Heisenberg algebras. Given a finite dimensional complex vector space $\mathcal{V}$ with an alternating bilinear form $\mathfrak{s}$ (if $\mathfrak{s}$ is nondegenerate, this is a symplectic form, whence the letter $\mathfrak{s}$) one can form the Heisenberg algebra $H_{\mathfrak{s}}:=T(\mathcal{V})/I_{\mathfrak{s}}$, where $T(\mathcal{V})=\bigoplus_{n=0}^{\infty}\mathcal{V}^{\otimes n}$ is the tensor algebra over $\mathcal{V}$ and $I_{\mathfrak{s}}$ is the ideal generated by elements of the form $v\otimes w-w\otimes v-\mathfrak{s}(v,w)$, so that $\mathfrak{s}$ becomes the commutator in the algebra $H_{\mathfrak{s}}$. It is clear, that $H_0=:H$ is the symmetric tensor algebra over $\mathcal{V}$ and it can be shown that the family $H_{t\mathfrak{s}}$, $t\in\mathbb{R}$ can be identified with a deformation of the symmetric algebra, i.e. there are invertible linear mappings $\Phi_t:H_{t\mathfrak{s}}\to H$ and we get a family $(\mu_t)_{t\in\mathbb{R}}$ of multiplcations on H (see [10] and references therein). Setting in particular $\mathcal{V}$ the vectorspace with basis $\{a, a^\dagger\}$ and $\mathfrak{s}(a, a^\dagger) = \hbar$ the obtained algebra is the algebra of the quantum harmonic oscillator. In [7] Majid just calls this a bialgebra like structure. It is in fact an example of an additive deformation in the following sense.

An additive deformation of a bialgebra $\mathcal{B}$ is a familiy $(\mu_t)_{t\in\mathbb{R}}$ of multiplications, which turns $\mathcal{B}_t = (\mathcal{B}, \mu_t, \mathbb{1})$ into a unital algebra ($\mathbb{1}$ is the unit element of the original algebra $\mathcal{B}$) such that $\Delta : \mathcal{B}_{t+s} \to \mathcal{B}_t \otimes \mathcal{B}_s$ is an algebra homomorphism and which satisfies some continuity condition (see Definition 1). Additive deformations are in 1-1-correspondence to commuting normalized 2-cocycles in the Hochschild cohomology associated with the $\mathcal{B}$-Bimodule structure on $\mathbb{C}$ given by the counit, i.e. $a.\lambda.b = \delta(a)\lambda\delta(b)$. Here a cocycle L is called commuting, if $L \star \mu = \mu \star L$ and it is called normalized, if $L(\mathbb{1} \otimes \mathbb{1}) = 0$ (see Theorem 1).

Wirth also showed in [10], that a Schoenberg correspondence holds for additive deformations. In [8] and [3] quantum Lévy processes on additive deformations are introduced, so additive deformations are of interest in quantum probability.

In the present paper we have two goals. First we wish to introduce a cohomology, such that we have a 1-1-correspondence between additive deformations and all cocycles. This also gives a concept of trivial deformations, i.e. deformations generated by a coboundary. We give a characterization of these trivial deformations. The second goal is to describe additive deformations of Hopf algebras. When one starts with a Hopf algebra, one would expect the deformed multiplications to have some analogue to the antipode, which we call deformed antipodes. We prove the existence of such deformed antipodes and describe their behaviour.

In Section 3 we introduce a cohomology, such that the generators of additive deformations are exactly the 2-cocycles. This is done by requiring each n-cochain to be normalized and to commute with $\mu^{(n)}$, the multiplication map for n factors. One has to show that this is a cochain complex, explicitly, that ∂c is normalized and commuting if c is. The same can be done for $*$-deformations of $*$-algebras.

Once the cohomology is established the question is, what kind of deformations are generated by coboundaries. It is shown, that those deformations are of the form

$$\mu_t = \Phi_t \circ \mu \circ (\Phi_t^{-1} \otimes \Phi_t^{-1})$$

where the Φ_t constitute a pointwise continuous one parameter group of invertible linear operators on $\mathcal{B}$ that commute in the sense that

$$(\Phi_t \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Phi_t) \circ \Delta.$$

When $L = \partial\psi$ is the generator of the additive deformation, then $\Phi_t = (\mathrm{id} \otimes e_\star^{t\psi})$ is the one parameter group of operators.

The second section of the paper discusses additive deformations of Hopf algebras. Deforming the multiplication of a bialgebra $\mathcal{B}$ also gives a deformed convolution product for linear maps from $\mathcal{B}$ to $\mathcal{B}$:

$$A \star_t B := \mu_t \circ (A \otimes B) \circ \Delta,$$

where $(\mu_t)_{t\in\mathbb{R}}$ is a deformation of the multiplication map μ of $\mathcal{B}$. If $\mathcal{B}$ is a Hopf algebra, i.e. there is an antipode S, which is the convolution inverse of the identity on $\mathcal{B}$ w.r.t. $\star = \mu \circ (\cdot \otimes \cdot) \circ \Delta$, it is quite natural to ask, whether there are also deformed antipodes S_t, which are convolution inverse to the identity map w.r.t. $\star_t$ and if they exist, what properties they have. Such a deformation is called a Hopf deformation.

In a Hopf algebra the antipode S is an algebra antihomomorphism and a coalgebra antihomomorphism, i.e.

$$S \circ \mu = \mu \circ (S \otimes S) \circ \tau$$
$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta.$$

Similar properties hold for the deformed antipodes S_t of a Hopf deformation. We can prove

$$S_t \circ \mu_{-t} = \mu_t \circ \tau \circ (S_t \otimes S_t) \tag{1}$$
$$\Delta \circ S_{t+r} = (S_t \otimes S_r) \circ \tau \circ \Delta. \tag{2}$$

Applying $\delta \otimes \delta$ to (2) we get

$$\delta \circ S_{t+r} = ((\delta \circ S_t) \otimes (\delta \circ S_r)) \circ \tau \circ \Delta = ((\delta \circ S_r) \otimes (\delta \circ S_t)) \circ \Delta,$$

i.e. $\delta \circ S_t$ is a convolution semigroup w.r.t. $\star = (\cdot \otimes \cdot) \circ \Delta$. So one would like to prove, that this semigroup has a generator, so that the S_t are of the form

$$S_t = S \star e_\star^{-t\sigma}. \tag{3}$$

To get a hint, how to find σ, we assume for the moment, that $\delta \circ S_t$ is differentiable in 0 and define

$$\sigma := -\left.\frac{\mathrm{d}}{\mathrm{d}t}\delta \circ S_t\right|_{t=0}.$$

Then we can apply δ to (3) and differentiate to get

$$L \circ (S \otimes \mathrm{id}) \circ \Delta - \sigma = L \circ (\mathrm{id} \otimes S) \circ \Delta - \sigma = 0$$

or after rearranging

$$\sigma = L \circ (S \otimes \mathrm{id}) \circ \Delta = L \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{4}$$

In fact we will prove, that every additive deoformation of a Hopf algebra is a Hopf deformation and (3) and (4) give a formula for the deformed antipodes.

In two special cases the structure can even be better understood. In the case of a trivial deformation it is easy to see, that

$$S_t = \Phi_t \circ S \circ \Phi_t$$

is another way to find the deformed antipodes. Differentiating this also gives a second formula for the generator

$$\sigma = -\psi - \psi \circ S.$$

If the bialgebra $\mathcal{B}$ is cocommutative we show, that every additive deformation splits in a trivial part and a part with constant antipodes. Applying δ to (1) and differentiating yields

$$-\sigma \circ \mu - L = L \circ (S \otimes S) \circ \tau - \sigma \otimes \delta - \delta \otimes \sigma$$

or after rearranging

$$L + L \circ (S \otimes S) \circ \tau = \delta \otimes \sigma - \sigma \circ \mu + \sigma \otimes \delta = \partial\sigma.$$

So L can be written as

$$L = \underbrace{\frac{1}{2}\partial\sigma}_{:=L_1} + \underbrace{\frac{1}{2}(L - L \circ (S \otimes S) \circ \tau)}_{:=L_2}$$

and if $\mathcal{B}$ is cocommutative the second part corresponds to constant antipodes.

2 Notation and Basic Definitions

All vector spaces considered are over the complex numbers, denoted by $\mathbb{C}$. The algebraic dual of a vector space $\mathcal{V}$ is denoted $\mathcal{V}' := \{\varphi : \mathcal{V} \to \mathbb{C} \mid \varphi \text{ linear}\}$. The tensor product $\otimes$ is the usual algebraic tensor product. If $\mathcal{V}$ is a vector space we write

$$\mathcal{V}^{\otimes n} := \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{n\times}$$

for $n \geq 1$ and $\mathcal{V}^{\otimes 0} := \mathbb{C}$.

A bialgebra $(\mathcal{B}, \mu, \mathbb{1}, \Delta, \delta)$ is a complex unital associative algebra $(\mathcal{B}, \mu, \mathbb{1})$ for which the mappings $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ and $\delta : \mathcal{B} \to \mathbb{C}$ are algebra homomorphisms and satisfy coassociativity and counit property respectively. A Hopf algebra is a bialgebra with an antipode, i.e. a linear mapping $S : \mathcal{B} \to \mathcal{B}$ with

$$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \mathbb{1}\delta = \mu \circ (S \otimes \mathrm{id}) \circ \Delta.$$

A $*$-bialgebra is a bialgebra with an involution, i.e. $(\mathcal{B}, \mu, \mathbb{1}, *)$ is a $*$-algebra and Δ, δ are $*$-homomorphisms. If $\mathcal{B}$ is a $*$-bialgebra, an involution on $\mathcal{B} \otimes \mathcal{B}$ is given by $(a \otimes b)^* = a^* \otimes b^*$. A $*$-Hopf algebra is a Hopf algebra which also is a $*$-bialgebra.

We use Sweedler's notation, writing $\Delta a = \sum_{k=0}^{n} a_k^{(1)} \otimes a_k^{(2)} =: a_{(1)} \otimes a_{(2)}$ and the notations $\mu^{(n)} : \mathcal{B}^{\otimes n} \to \mathcal{B}$, $\Delta^{(n)} : \mathcal{B} \to \mathcal{B}^{\otimes n}$

$$\mu^{(0)}(\lambda) = \lambda \mathbb{1} \qquad \Delta^{(0)} = \delta$$
$$\mu^{(n+1)} = \mu \circ (\mathrm{id} \otimes \mu^{(n)}) \qquad \Delta^{(n+1)} = (\mathrm{id} \otimes \Delta^{(n)}) \circ \Delta.$$

The Sweedler notation for this is

$$\Delta^{(n)} a = a_{(1)} \otimes \cdots \otimes a_{(n)}.$$

With $\mathcal{B}$ also each $\mathcal{B}^{\otimes n}$ is a bialgebra in the natural way. We frequently use the comultiplication on $\mathcal{B} \otimes \mathcal{B}$, which we denote by Λ and which is defined by

$$\Lambda(a \otimes b) = a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)},$$

i.e. $\Lambda = (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$. The counit of $\mathcal{B} \otimes \mathcal{B}$ is just $\delta \otimes \delta$.

If $(\mathcal{C}, \Delta)$ is a coalgebra and $(\mathcal{A}, m)$ is an algebra we define the convolution product for mappings $R, S : \mathcal{C} \to \mathcal{A}$ by $R \star S := m \circ (R \otimes S) \circ \Delta$. In our context $\mathcal{C}$ and $\mathcal{A}$ are usually tensor powers of the same bialgebra $\mathcal{B}$.

A pointwise continuous convolution semigroup is a family $(\varphi_t)_{t \geq 0}$ of linear mappings $\varphi_t : \mathcal{B} \to \mathbb{C}$ such that

- $\varphi_t \star \varphi_s = \varphi_{t+s}$
- $\varphi_t(b) \xrightarrow{t \to 0} \delta(b) \quad \forall b \in \mathcal{B}$

Note that δ is the unit for the multiplication $\star$ on $\mathcal{B}'$ (This is exactly the counit property). It follows from the fundamental theorem for coalgebras, that for a pointwise continuous convolution semigroup there exists a generator ψ, which is the pointwise limit

$$\psi(b) = \left.\frac{\mathrm{d}\varphi_t(b)}{\mathrm{d}t}\right|_{t=0} = \lim_{t \to 0} \frac{\varphi_t(b) - \delta(b)}{t}$$

and for which we have

$$\varphi_t = e_\star^{t\psi}.$$

Cf. [2] section 4 for details.

Definition 1. An *additive deformation* of the bialgebra $\mathcal{B}$ is a family $(\mu_t)_{t \geq 0}$ of mappings $\mu_t : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ such that

1. for each $t \geq 0$ $(\mathcal{B}, \mu_t, \mathbb{1})$ is an unital algebra
2. $\mu_0 = \mu$
3. $\Delta \circ \mu_{t+s} = (\mu_t \otimes \mu_s) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$ (τ denotes the flip on $\mathcal{B} \otimes \mathcal{B}$)
4. the mapping $t \mapsto \delta \circ \mu_t$ is pointwise continuous, i.e.$\delta \circ \mu_t \xrightarrow{t \to 0} \delta \circ \mu = \delta \otimes \delta$ pointwise
5. if $\mathcal{B}$ is a $*$-bialgebra and for each $t \geq 0$ $(\mathcal{B}, \mu_t, \mathbb{1}, *)$ is a unital $*$-algebra, we call the deformation an additive deformation of a $*$-bialgebra.

Theorem 1 (Cf. [10]). *Let $(\mu_t)_{t\geq 0}$ be an additive Deformation of the bialgebra $\mathcal{B}$. Then* $L = \left.\frac{\mathrm{d}(\delta\circ\mu_t)}{\mathrm{d}t}\right|$ *exists pointwise and we have for $a, b, c \in \mathcal{B}, t \geq 0$*

1. $\mu_t = \mu \star e_\star^{tL}$
2. $\mu \star L = L \star \mu$ *'L is commuting'*
3. $L(\mathbb{1} \otimes \mathbb{1}) = 0$ *'L is normalized'*
4. $\delta(a)L(b \otimes c) - L(ab \otimes c) + L(a \otimes bc) - L(a \otimes b)\delta(c) = 0.$

If $(\mu_t)_{t\geq 0}$ is a $*$-bialgebra deformation, then

5. $L(b \otimes c) = \overline{L(c^* \otimes b^*)}$ *'L is hermitian'*

also holds.

Conversely, if $L : \mathcal{B} \otimes \mathcal{B} \to \mathbb{C}$ is a linear mapping, which fulfills conditions 2,3 and 4 (in case of $*$-bialgebra also 5), than the first equation defines an additive deformation on $\mathcal{B}$.

3 Cohomology of Additive Deformations

3.1 Subcohomologies of the Hochschild cohomology

A cochain complex (ccc) consists of a sequence of vector spaces $C = (C_n)_{n\in\mathbb{N}}$ and linear mappings $\partial_n : C_n \to C_{n+1}$ such that $\partial_{n+1} \circ \partial_n = 0$ for all $n \in \mathbb{N}$. The elements of $Z_n(C) = \operatorname{kern} \partial_n$ are called *(n−)cocycles*, the elements of $B_n(C) = \operatorname{im} \partial_{n-1}$ are called *(n−)coboundaries* and the vector-space $H_n(C) = Z_n(C)/B_n(C)$ is called *n-th cohomology.* A sequence $D = (D_n)_{n\in\mathbb{N}}$ is called sub-ccc, if $D_n \subseteq C_n$ and $\partial_n D_n \subseteq D_{n+1}$ for all n. Then $\big((D_n)_{n\in\mathbb{N}}, (\partial_n|_{D_n})_{n\in\mathbb{N}}\big)$ is again a ccc and we have:

1. The cocycles of D are exactly the cocycles of C, belonging to D, i.e.
$$Z_n(D) = Z_n(C) \cap D_n,$$
2. each coboundary of D is a coboundary of C, i.e.
$$B_n(D) \subseteq B_n(C) \cap D_n,$$
3. equality holds, iff the mapping $H_n(D) \to H_n(C), f + B_n(D) \mapsto f + B_n(C)$ is an injection,
4. If D, E are sub-ccc's, then $(D_n \cap E_n)_{n\in\mathbb{N}}$ is a sub-ccc.

Points 1,2 and 4 are obvious, while 3 follows from the observation, that the kernel of the given mapping is exactly $B_n(C) \cap D_n$.

For an algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule M we define

$$C_n = \operatorname{Lin}(\mathcal{A}^{\otimes n}, M)$$

One can show, that together with the coboundary-operator

$$\partial_n f(a_1, \ldots, a_{n+a}) :=$$
$$a_1.f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i\, f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1}\, f(a_1, \ldots, a_n).a_{n+1}$$

this is a ccc. Especially for $\mathcal{A} = \mathcal{B}$ a bialgebra and $M = \mathbb{C}$ the $\mathcal{B}$-bimodule given by $a.\lambda.b = \delta(a)\lambda\delta(b)$ for $\lambda \in \mathbb{C}$ and $a, b \in \mathcal{B}$ we have

$$\partial_n f(a_1, \dots, a_{n+a}) := \delta(a_1) f(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \ f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ + (-1)^{n+1} \ f(a_1, \dots, a_n)\delta(a_{n+1}). \quad (5)$$

The generators of additive deformations are normalized commuting cocycles, so it is natural to define

$$C_n^{(\mathbf{N})} = \left\{ f \in C_n \mid f(\mathbb{1}^{\otimes n}) = 0 \right\}$$
$$C_n^{(\mathbf{C})} = \left\{ f \in C_n \mid f \star \mu^{(n)} = \mu^{(n)} \star f \right\}$$

If $\mathcal{B}$ is a $*$-bialgebra the generators are also hermitian. We define for $f \in C_n$

$$\widetilde{f}(a_1 \otimes \dots \otimes a_n) := \overline{f(a_n^* \otimes \dots \otimes a_1^*)}$$

and set

$$C_n^{(\mathbf{H})} = \begin{cases} \left\{ f \in C_n \;\middle|\; \widetilde{f} = f \right\}, & \text{if } \lceil \frac{n}{2} \rceil \text{ odd, i.e. } n = 1, 2, 5, 6, \dots \\ \left\{ f \in C_n \;\middle|\; \widetilde{f} = -f \right\}, & \text{if } \lceil \frac{n}{2} \rceil \text{ even, i.e. } n = 0, 3, 4, 7, 8, \dots \end{cases}$$

Proposition. $C_n^{(\mathbf{N})}$, $C_n^{(\mathbf{C})}$ and $C_n^{(\mathbf{H})}$ are sub-ccc's of C_n.

Proof. We only need to show, that $\partial C_n^{(*)} \subseteq C_n^{(*)}$ for $* = \mathbf{N}, \mathbf{C}, \mathbf{H}$.

N: Let $f \in C_n^{(\mathbf{N})}$, i.e. $f(\mathbb{1}^{\otimes n}) = 0$. Then

$$\partial f(\mathbb{1}^{\otimes(n+1)}) = \delta(\mathbb{1}) f(\mathbb{1}^{\otimes n}) + \sum_{i=1}^{n} (-1)^i \ f(\mathbb{1}^{\otimes n}) + (-1)^{n+1} \ f(\mathbb{1}^{\otimes n})\delta(\mathbb{1}) = 0$$

C: For $f \in C_n^{(\mathbf{C})}$ we get

$$\partial f \star \mu^{(n+1)} = \\ \left(\delta \otimes f + \sum_{k=1}^{n} (-1)^k f \circ (\mathrm{id}_{k-1} \otimes \mu \otimes \mathrm{id}_{n-k}) + (-1)^{n+1} f \otimes \delta \right) \star \mu^{(n+1)}.$$

Next we show, that each summand commutes with μ under convolution:

$$\begin{aligned} &(\delta \otimes f) \star \mu^{(n+1)}(a_1 \otimes \dots \otimes a_{n+1}) \\ &= \delta(a_1^{(1)}) f\left(a_2^{(1)} \otimes \dots \otimes a_{n+1}^{(1)}\right) a_1^{(2)} \dots a_{n+1}^{(2)} \\ &= a_1 f\left(a_2^{(1)} \otimes \dots \otimes a_{n+1}^{(1)}\right) a_2^{(2)} \dots a_{n+1}^{(2)} \\ &= a_1 f\left(a_2^{(2)} \otimes \dots \otimes a_{n+1}^{(2)}\right) a_2^{(1)} \dots a_{n+1}^{(1)} \qquad \text{as } f \star \mu^{(n)} = \mu^{(n)} \star f \\ &= a_1^{(1)} \dots a_{n+1}^{(1)} \delta(a_1^{(2)}) f\left(a_2^{(2)} \otimes \dots \otimes a_{n+1}^{(2)}\right) \\ &= \mu^{(n+1)} \star (\delta \otimes f)(a_1 \otimes \dots \otimes a_{n+1}). \end{aligned}$$

Analoguesly we see that

$$(f \otimes \delta) \star \mu^{(n+1)} = \mu^{(n+1)} \star (f \otimes \delta).$$

For the remaining summands we calculate

$$
\begin{aligned}
&(f\circ(\mathrm{id}_{k-1}\otimes\mu\otimes\mathrm{id}_{n-k}))\star\mu^{(n+1)}(a_1\otimes\cdots\otimes a_{n+1})\\
&=f\left(a_1^{(1)}\otimes\cdots\otimes(a_k^{(1)}a_{k+1}^{(1)})\otimes\cdots\otimes a_{n+1}^{(1)}\right)a_1^{(2)}\ldots a_k^{(2)}a_{k+1}^{(2)}\ldots a_{n+1}^{(2)}\\
&=f\left(a_1^{(1)}\otimes\cdots\otimes(a_ka_{k+1})^{(1)}\otimes\cdots\otimes a_{n+1}^{(1)}\right)a_1^{(2)}\ldots(a_ka_{k+1})^{(2)}\ldots a_{n+1}^{(2)}
\end{aligned}
$$

(as Δ is an algebra-homomorphism)

$$
=f\left(a_1^{(2)}\otimes\cdots\otimes(a_ka_{k+1})^{(2)}\otimes\cdots\otimes a_{n+1}^{(2)}\right)a_1^{(1)}\ldots(a_ka_{k+1})^{(1)}\ldots a_{n+1}^{(1)}
$$

(as $f\star\mu^{(n)}=\mu^{(n)}\star f$)

$$
=\mu^{(n+1)}\star(f\circ(\mathrm{id}_{k-1}\otimes\mu\otimes\mathrm{id}_{n-k}))(a_1\otimes\cdots\otimes a_{n+1}).
$$

H: Let $\widetilde{f}=\pm f$. For n odd, we get

$$
\begin{aligned}
&\widetilde{\partial f}(a_1,\ldots,a_{n+1})=\overline{\partial f(a_{n+1}^*,\ldots,a_1^*)}=\overline{\delta(a_{n+1}^*)f(a_n^*,\ldots,a_1^*)}\\
&\overline{+\sum_{i=1}^{n}(-1)^{n+1-i}\ f(a_{n+1}^*,\ldots,a_{i+1}^*a_i^*,\ldots,a_1^*)+f(a_{n+1}^*,\ldots,a_2^*)\delta(a_1^*)}\\
&=\delta(a_1)\widetilde{f}(a_2,\ldots,a_{n+1})+\sum_{i=1}^{n}(-1)^i\ \widetilde{f}(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1})+\widetilde{f}(a_1,\ldots,a_n)\delta(a_{n+1})\\
&=\pm\partial f(a_1,\ldots,a_{n+1})
\end{aligned}
$$

and for n even

$$
\begin{aligned}
&\widetilde{\partial f}(a_1,\ldots,a_{n+1})=\overline{\partial f(a_{n+1}^*,\ldots,a_1^*)}=\overline{\delta(a_{n+1}^*)f(a_n^*,\ldots,a_1^*)}\\
&\overline{+\sum_{i=1}^{n}(-1)^{n+1-i}\ f(a_{n+1}^*,\ldots,a_{i+1}^*a_i^*,\ldots,a_1^*)-f(a_{n+1}^*,\ldots,a_2^*)\delta(a_1^*)}\\
&=-\delta(a_1)\widetilde{f}(a_2,\ldots,a_{n+1})-\sum_{i=1}^{n}(-1)^i\ \widetilde{f}(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1})+\widetilde{f}(a_1,\ldots,a_n)\delta(a_{n+1})\\
&=\mp\partial f(a_1,\ldots,a_{n+1}).
\end{aligned}
$$

□

Since the intersection of sub-ccc's is again a sub-ccc we have

Corollary. *$C_n^{(\mathbf{NC})}:=C_n^{(\mathbf{N})}\cap C_n^{(\mathbf{C})}$ and $C_n^{(\mathbf{NCH})}:=C_n^{(\mathbf{NC})}\cap C_n^{(\mathbf{H})}$ are ccc's with the coboundary-operator* (5)*.*

3.2 Characterization of the trivial deformations

For an additive deformation of the bialgebra $\mathcal{B}$ the generator L of the convolution-semigroup $(\delta\circ\mu_t)_{t\geq 0}$ is an element of $Z_2^{(\mathbf{NC})}$ and conversely if $L\in Z_2^{(\mathbf{NC})}$ we can define an additive deformation via $\mu_t:=\mu\star e_\star^{tL}$. In the case of a $*$-Bialgebra the generators are exactly the elements of $Z_2^{(\mathbf{NCH})}$. We wish to answer the question which deformations are generated by the coboundaries, i.e. the elements of $B_2^{(\mathbf{NC})}$ or $B_2^{(\mathbf{NCH})}$ respectively.

Theorem 2. *Let $\mathcal{B}$ be a bialgebra, $L \in B_2^{(\mathbf{NC})}$, $L = \partial\psi$ with $\phi \in C_1^{(\mathbf{NC})}$. Then $(\Phi_t)_{t\geq 0}$ is a pointwise continuous semigroup of unital algebra isomorphisms $\Phi_t : (\mathcal{B}, \mu_t) \to (\mathcal{B}, \mu)$, for which*

$$(\Phi_t \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Phi_t) \circ \Delta \quad \textit{for all } t \geq 0, \tag{6}$$

where $\Phi_t = \mathrm{id} \star e_\star^{t\psi}$ and $\mu_t = \mu \star e_\star^{tL}$. If $\mathcal{B}$ is a $*$-algebra and $L \in B_2^{(\mathbf{NCH})}$, then we can choose $\psi \in C_1^{(\mathbf{NCH})}$ and the Φ_t are $*$-algebra isomorphisms.

Conversely, if $(\Phi_t)_{t\geq 0}$ is a pointwise continuous semigroup of invertible linear mappings $\Phi_t : \mathcal{B} \to \mathcal{B}$, such that $\Phi_t(\mathbb{1}) = \mathbb{1}$ for all $t \geq 0$, and which fulfills (6), then

$$\mu_t := \Phi_t^{-1} \circ \mu \circ (\Phi_t \otimes \Phi_t)$$

defines an additive Deformation of $\mathcal{B}$ with generator $L \in B_2^{(\mathbf{NC})}$. If $\mathcal{B}$ is a $*$-algebra and the Φ_t are hermitian, then we get an additive deformation of a $*$-bialgebra and $L \in B_2^{(\mathbf{NCH})}$.

Before we prove this, we recall the following Lemma. When $\mathcal{B}$ is a bialgebra and $\varphi : \mathcal{B} \to \mathbb{C}$ a linear functional on $\mathcal{B}$ we define

$$R_\varphi : \mathcal{B} \to \mathcal{B}, \quad R_\varphi := \mathrm{id} \star \varphi = (\mathrm{id} \otimes \varphi) \circ \Delta.$$

Lemma 1. *For $\varphi, \psi \in \mathcal{B}'$ the following holds:*

1. $R_\varphi \circ R_\psi = R_{\varphi\star\psi}$
2. $\delta \circ R_\varphi = \varphi$
3. $R_{\delta\otimes\varphi} = \mathrm{id} \otimes R_\varphi$
4. $R_{\varphi\otimes\delta} = R_\varphi \otimes \mathrm{id}$
5. $\mu \circ R_{\varphi\circ\mu} = R_\varphi \circ \mu$

Note that the last three equations are between operators on the bialgebra $\mathcal{B} \otimes \mathcal{B}$.

Proof. This is all straightforward to verify. □

Proof of Theorem 2. Let $\mathcal{B}$ be a bialgebra and $L \in B_2^{(\mathbf{NC})}$ a coboundary, $L = \partial\psi$ with $\psi \in C_1^{(\mathbf{NC})}$. We write $\varphi_t := e_\star^{t\psi}$ and note, that this is a pointwise continuous convolution semigroup and the φ_t are commuting (i.e. $\varphi_t \star \mathrm{id} = \mathrm{id} \star \varphi_t$) since ψ is. Then the mappings $\Phi_t = R_{\varphi_t}$ yield a pointwise continuous semigroup of linear operators on $\mathcal{B}$ and we only need to show, that they are unital algebra isomorphisms. It is obvious, that $\Phi_t(\mathbb{1}) = \mathbb{1}$, since $\psi(\mathbb{1}) = 0$, and $\Phi_t \circ \Phi_{-t} = \mathrm{id}$, so Φ_t is invertible. We have to prove that $\Phi_t : (\mathcal{B}, \mu) \to (\mathcal{B}, \mu_t)$ is an algebra homomorphism, i.e.

$$\mu_t = \mu \star e_\star^{tL} = \Phi_t^{-1} \circ \mu \circ (\Phi_t \otimes \Phi_t).$$

From

$$\begin{aligned} e_\star^{tL} &= e_\star^{t\partial\psi} \\ &= e_\star^{t(\delta\otimes\psi - \psi\circ\mu + \psi\otimes\delta)} \\ &= e_\star^{-t\psi\circ\mu} \star e_\star^{t\delta\otimes\psi} \star e_\star^{t\psi\otimes\delta} \\ &= (\varphi_{-t} \circ \mu) \star (\delta \otimes \varphi_t) \star (\varphi_t \otimes \delta), \end{aligned}$$

where we used that $\delta \otimes \psi$, $\psi \circ \mu$ and $\psi \otimes \delta$ commute under convolution, we conclude

$$\begin{aligned} \mu_t &= \mu \star e_\star^{t\partial\psi} \\ &= (\mu \otimes e_\star^{t\partial\psi}) \circ \Lambda \\ &= \mu \circ R_{e_\star^{t\partial\psi}} \\ &= \mu \circ R_{(\varphi_{-t}\circ\mu)\star(\delta\otimes\varphi_t)\star(\varphi_t\otimes\delta)} \\ &= \mu \circ R_{\varphi_{-t}\circ\mu} \circ (\mathrm{id} \otimes R_{\varphi_t}) \circ (R_{\varphi_t} \otimes \mathrm{id}) \\ &= R_{\varphi_{-t}} \circ \mu \circ (R_{\varphi_t} \otimes R_{\varphi_t}) \\ &= \Phi_t^{-1} \circ \mu \circ (\Phi_t \otimes \Phi_t). \end{aligned}$$

It is clear that the Φ_t are $*$-homomorphisms in the $*$-bialgebra case.

Now let $(\Phi_t)_{t\geq 0}$ be pointwise continuous semigroup of invertible linear mappings with $\Phi_t(\mathbb{1}) = \mathbb{1}$ and $(\Phi_t \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Phi_t) \circ \Delta$. Then we write $\varphi_t = \delta \circ \Phi_t$ and observe that

1. $(\varphi_t)_{t\geq 0}$ is a pointwise continuous convolution semigroup, so that there is a $\psi \in C_1^{(\mathbf{NC})}$ with $\varphi_t = e_\star^{t\psi}$. Indeed

$$\begin{aligned}\varphi_t \star \varphi_s &= ((\delta \circ \Phi_t) \otimes (\delta \circ \Phi_s)) \circ \Delta \\ &= (\delta \otimes \delta) \circ (\Phi_t \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Phi_s) \circ \Delta \\ &= (\delta \otimes \delta) \circ (\mathrm{id} \otimes \Phi_t) \circ (\Phi_s \otimes \mathrm{id}) \circ \Delta \\ &= (\delta \otimes \delta) \circ (\mathrm{id} \otimes \Phi_{t+s}) \circ \Delta \\ &= \varphi_{t+s}\end{aligned}$$

and $\psi(\mathbb{1}) = 0, \psi \star \mathrm{id} = \mathrm{id} \star \psi$ follow from $\varphi_t(\mathbb{1}) = 1$ and $\varphi_t \star \mathrm{id} = \mathrm{id} \star \varphi_t$ via differentiation. If the Φ_t are hermitian, ψ is also hermitian, i.e. $\psi \in C_1^{(\mathbf{NCH})}$.

2. $\Phi_t = R_{\varphi_t}$, as

$$\begin{aligned}R_{\varphi_t} &= (\mathrm{id} \otimes (\delta \circ \Phi_t)) \circ \Delta \\ &= (\mathrm{id} \otimes \delta) \circ (\mathrm{id} \otimes \Phi_t) \circ \Delta \\ &= (\mathrm{id} \otimes \delta) \circ (\Phi_t \otimes \mathrm{id}) \circ \Delta \\ &= \Phi_t.\end{aligned}$$

So the first part of the theorem tells us, that $L = \partial\psi \in B_2^{(\mathbf{NC})}$ is the generator of an additive deformation, for which

$$\mu_t = \Phi_t^{-1} \circ \mu \circ (\Phi_t \otimes \Phi_t).$$

If B is a $*$-algebra and all the Φ_t are hermitian, then so are all the φ_t and via differentiation also ψ. That means $L \in B_2^{(\mathbf{NCH})}$ and the deformation is a deformation of $*$-algebras. □

4 Additive Deformations of Hopf Algebras

4.1 Definition of Hopf deformations and general observations

Lemma 2. *Let $\mathcal{B}$ be a Bialgebra and L generator of an additive deformation. Then we can define*

$$\mu_t := e_\star^{tL} \star \mu$$

for all $t \in \mathbb{R}$ (i.e. not only for $t \geq 0$) and

$$\Delta : \mathcal{B}_{t+s} \to \mathcal{B}_t \otimes \mathcal{B}_s$$

is an algebra homomorphism for all $s, t \in \mathbb{R}$.

Proof. It follows from Theorem 1 that $-L$ is the generator of an additive deformation, so for $t < 0$ the definition of μ_t yields a multiplication on $\mathcal{B}$. We calculate

$$\begin{aligned}\Delta \circ \mu_{t+s} &= \Delta \circ (\mu \otimes e_\star(t+s)L) \circ \Lambda \\ &= ((\Delta \circ \mu) \otimes e_\star(t+s)L) \circ \Lambda \\ &= (\mu \otimes \mu \otimes e_\star^{tL} \otimes e_\star^{sL}) \circ \Lambda^{(4)} \\ &= (\mu \otimes e_\star^{tL} \otimes \mu \otimes e_\star^{sL}) \circ \Lambda^{(4)} \\ &= ((\mu \star e_\star tL) \otimes (\mu \star e_\star sL)) \circ \Lambda \\ &= (\mu_t \otimes \mu_s) \circ \Lambda\end{aligned}$$

□

From now on we always view an additive deformation as a family of multiplications indexed by all real numbers.

Definition 2. An additive deformation is called a *Hopf deformation*, if for all $t \in \mathbb{R}$ there exists a linear mapping $S_t : \mathcal{B} \to \mathcal{B}$ such that

$$\mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta = \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta = \delta \mathbb{1}. \tag{7}$$

For $t = 0$ this of course implies, that $\mathcal{B}$ is a Hopf algebra with antipode $S = S_0$.

Many proofs in this section follow a common path. To show an identity $a = b$, we find an element c and a convolution product $\diamond$ such that $a \diamond c = c \diamond b = \delta$ where δ is the neutral element for $\diamond$. Then we conclude

$$a = a \diamond \delta = a \diamond c \diamond b = \delta \diamond b = b$$

and hence

$$a = b = c^{-1}$$

Let $\mathcal{B}$ be a bialgebra with additive deformation $(\mu_t)_{t\in\mathbb{R}}$ and $\mu_t = \mu \star e_\star^{tL}$ for a normalized, commuting cocycle L.

Theorem 3. *If a family S_t with* (7) *exists, it is uniquely determined and the following statements hold:*

1. $S_t(\mathbb{1}) = \mathbb{1}$
2. $S_t : B_{-t} \to B_t$ *is an algebra antihomomorphism, i.e.*

$$S_t \circ \mu_{-t} = \mu_t \circ (S_t \otimes S_t) \circ \tau \tag{8}$$

3. $\Delta \circ S_{t+r} = (S_t \otimes S_r) \circ \tau \circ \Delta$
4. *If B is cocommutative, i.e. $\Delta = \tau \circ \Delta$, then*

$$S_t \circ S_{-t} = \mathrm{id}$$

for all $t \in \mathbb{R}$

Proof. **(Uniqueness)** The uniqueness statement is clear, as (7) states, that S_t is the two-sided convolution inverse of the identity mapping on $\mathcal{B}$ w.r.t. $\star_t$.

1. This is clear, since

$$\mathbb{1} = \mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta(\mathbb{1}) = S_t(\mathbb{1}).$$

2. We show, that both sides of (8) are convolution inverses of μ_t w.r.t. $\star_t$:

$$\begin{aligned}(S_t \circ \mu_{-t}) \star_t \mu_t &= \mu_t \circ (S_t \otimes \mathrm{id}) \circ (\mu_{-t} \otimes \mu_t) \circ \Lambda \\ &= \mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta \circ \mu = \delta \circ \mu \mathbb{1} = \delta \otimes \delta \mathbb{1}\end{aligned}$$

and

$$\begin{aligned}&\mu_t \star_t (\mu_t \circ (S_t \otimes S_t) \circ \tau)(a \otimes b) \\ &= \mu_t \circ (\mu_t \otimes \mu_t) \circ (\mathrm{id}_2 \otimes ((S_t \otimes S_t) \circ \tau)) \circ \Lambda (a \otimes b) \\ &= \mu_t^{(4)}(a_{(1)} \otimes b_{(1)} \otimes S_t(b_{(2)}) \otimes S_t(a_{(2)})) \\ &= \delta(b) \mu_t(a_{(1)} \otimes S_t(a_{(1)})) \\ &= \delta(a)\delta(b)\mathbb{1}\end{aligned}$$

3. For linear maps from the coalgebra (B,Δ) to the algebra $(B_t\otimes B_r)$ we have a convolution $\diamond$ defined as

$$A\diamond B=(\mu_t\otimes\mu_r)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(A\otimes B)\circ\Delta.$$

We show that both sides of (2) are inverses of Δ w.r.t. $\diamond$:

$$\begin{aligned}(\Delta\circ S_{t+r})\diamond\Delta&=(\mu_t\otimes\mu_r)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta)\circ(S_{t+r}\otimes\mathrm{id})\circ\Delta\\&=\Delta\circ\mu_{t+r}\circ(S_{t+r}\otimes\mathrm{id})\circ\Delta=\delta\,\Delta(\mathbb{1})=\delta\mathbb{1}\otimes\mathbb{1}\end{aligned}$$

and

$$\begin{aligned}&\Delta\diamond((S_t\otimes S_r)\circ\tau\circ\Delta)(a)\\&=(\mu_t\otimes\mu_r)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{id}_2\otimes S_t\otimes S_r)\circ(\mathrm{id}_2\otimes\tau)\circ\Delta^{(4)}(a)\\&=(\mu_t\otimes\mu_r)(a_{(1)}\otimes S_t(a_{(4)})\otimes a_{(2)}\otimes S_r(a_{(3)}))\\&=\mu_t(a_{(1)}\otimes S_t(a_{(2)}))\otimes\mathbb{1}\\&=\delta(a)\mathbb{1}\otimes\mathbb{1}\end{aligned}$$

4. Let $\Delta=\tau\circ\Delta$. Then

$$\begin{aligned}(S_t\circ S_{-t})\star_t S_t&=\mu_t\circ(S_t\otimes S_t)\circ(S_{-t}\otimes\mathrm{id})\circ\Delta\\&=S_t\circ\mu_{-t}\circ\tau\circ(S_{-t}\otimes\mathrm{id})\circ\Delta\\&=S_t\circ\mu_{-t}\circ(\mathrm{id}\otimes S_{-t})\circ\tau\circ\Delta\\&=\delta S_t(\mathbb{1})=\delta\mathbb{1}\end{aligned}$$

□

4.2 The deformed antipodes for trivial deformations

Theorem 4. *Let $\mathcal{B}$ be a Hopf algebra and $(\mu_t)_{t\in\mathbb{R}}$ a trivial deformation,*

$$\mu_t=\Phi_t\circ\mu\circ(\Phi_t^{-1}\otimes\Phi_t^{-1}),$$

and

$$\Phi_t=\mathrm{id}\star e_\star^{t\psi}$$

for a commuting, normalized linear functional ψ. Then

$$S_t=\Phi_t\circ S\circ\Phi_t=S\star e_\star^{t(\psi\circ S+\psi)}$$

is the deformed antipode, so $(\mu_t)_{t\in\mathbb{R}}$ is a Hopf deformation.

Proof. All we have to show is that

$$\mu_t\circ(S_t\otimes\mathrm{id})\circ\Delta=\mu_t\circ(\mathrm{id}\otimes S_t)\circ\Delta=\delta\mathbb{1},$$

for $S_t=\Phi_t\circ S\circ\Phi_t$ and $S_t=S\star e_\star^{t(\psi\circ S+\psi)}$. In the case $S_t=\Phi_t\circ S\circ\Phi_t$ we calculate

$$\begin{aligned}\mu_t\circ(S_t\otimes\mathrm{id})\circ\Delta&=\Phi_t\circ\mu\circ(\Phi_t^{-1}\otimes\Phi_t^{-1})\circ((\Phi_t\circ S\circ\Phi_t)\otimes\mathrm{id})\circ\Delta\\&=\Phi_t\circ\mu\circ(S\circ\Phi_t\otimes\Phi_t^{-1})\circ\Delta\\&=\Phi_t\circ\mu\circ(S\otimes\Phi_t^{-1})\circ(\Phi_t\otimes\mathrm{id})\circ\Delta\\&=\Phi_t\circ\mu\circ(S\otimes\Phi_t^{-1})\circ(\mathrm{id}\otimes\Phi_t)\circ\Delta\\&=\Phi_t\circ\mu\circ(S\otimes\mathrm{id})\circ\Delta\\&=\delta\Phi_t(\mathbb{1})=\delta\mathbb{1}\end{aligned}$$

and the second equality is proved in the same way.

Now we consider the case $S_t = S \star e_\star^{t(\psi\circ S+\psi)}$. We first recall that ψ ist commuting and $L = -\partial\psi$ is the generator of the additive deformation. Next we observe that

$$\begin{aligned}(\psi\circ S)\star S &= (\psi\otimes \mathrm{id})\circ(S\otimes S)\circ\Delta\\ &= (\psi\otimes \mathrm{id})\circ\Delta^{op}\circ S\\ &= (\psi\otimes \mathrm{id})\circ\Delta\circ S\\ &= (\mathrm{id}\otimes\psi)\circ(S\otimes S)\circ\Delta\\ &= S\star(\psi\circ S).\end{aligned}$$

With this in mind we calculate

$$\begin{aligned}\mu_t\circ(S_t\otimes \mathrm{id})\circ\Delta\,(a) &= (\mu\otimes e_\star^{tL})\circ\Lambda(e_\star^{t\psi}(S(a_{(1)}))e_\star^{t\psi}(a_{(2)})S(a_{(3)})\otimes a_{(4)})\\ &= e_\star^{t\psi}(S(a_{(1)}))e_\star^{t\psi}(a_{(2)})e_\star^{tL}(S(a_{(3)})\otimes a_{(4)})\\ &= \delta(a)\end{aligned}$$

since

$$\begin{aligned}&e_\star^{tL}(S(a_{(1)})\otimes a_{(2)})\\ &= e_\star^{-t\psi\otimes\delta}(S(a_{(1)})\otimes a_{(2)})e_\star^{\psi\circ\mu}(S(a_{(3)})\otimes a_{(4)})e_\star^{-t\delta\otimes\psi}(S(a_{(5)})\otimes a_{(6)})\\ &= e_\star^{-t\psi}(S(a_{(1)}))e_\star^{-t\psi}(a_{(2)}).\end{aligned}$$

Again the second equality is proven similarly.

One can also prove this by writing $\Phi_t = (e_\star^{t\psi}\otimes \mathrm{id})\circ\Delta$ in $S_t = \Phi_t\circ S\circ\Phi_t$ and using that S, ψ and $\psi\circ S$ all commute with each other. □

It is still possible that the deformed antipodes are constant. We have

Theorem 5. *Let L be generator of a trivial additive deformation. Then it has constant antipodes, i.e. $S_t = S$ $\forall t\in\mathbb{R}$ iff*

$$\Phi_t\circ S = S\circ\Phi_{-t}.$$

for all $t\in\mathbb{R}$.

Proof. This follow directly from $S_t = \Phi_t\circ S\circ\Phi_t$ and $\Phi_t^{-1} = \Phi_{-t}$. □

4.3 The deformed antipodes in the general case

We want to show, that every additive deformation of a Hopf algebra is a Hopf deformation and give a formula for the deformed antipodes.

Lemma 3. *We have*

$$L\circ(S\otimes \mathrm{id})\circ\Delta = L\circ(\mathrm{id}\otimes S)\circ\Delta.$$

Proof. From $\partial L = 0$ it follows that

$$\begin{aligned}0 &= \partial L(a_{(1)}\otimes S(a_{(2)}\otimes a_{(3)})\\ &= \delta(a_{(1)})L\big(S(a_{(2)})\otimes a_{(3)}\big) - L\big(a_{(1)}S(a_{(2)})\otimes a_{(2)}\big)\\ &\quad + L\big(a_{(1)}\otimes S(a_{(2)}a_{(3)})\big) - L\big(a_{(1)}\otimes a_{(2)}\big)\delta(a_{(3)})\\ &= L(S(a_{(1)})\otimes a_{(2)}) - L(a_{(1)}\otimes S(a_{(2)}).\end{aligned}$$

□

Definition 3. Let $\mathcal{B}$ be a Hopf algebra and L generator of an additive deformation. Then we set

$$\sigma := L\circ(\mathrm{id}\otimes S)\circ\Delta.$$

Lemma 4. *σ is commuting, i.e.*

$$(\sigma\otimes \mathrm{id})\circ\Delta = (\mathrm{id}\otimes\sigma)\circ\Delta.$$

Proof. First we observe that

$$\begin{aligned} L(a_{(1)} \otimes S(a_{(2)})) &= L(a_{(1)} \otimes S(a_{(4)}))a_{(2)}S(a_{(3)}) \\ &= L \star \mu(a_{(1)} \otimes S(a_{(2)})) \\ &= \mu \star L(a_{(1)} \otimes S(a_{(2)})) \\ &= L(a_{(2)} \otimes S(a_{(3)}))a_{(1)}S(a_{(4)}). \end{aligned}$$

Now we calculate

$$\begin{aligned} (\sigma \otimes \mathrm{id}) \circ \Delta(a) &= \sigma(a_{(1)})a_{(2)} \\ &= (L \circ (\mathrm{id} \otimes S) \circ \Delta)(a_{(1)})\, a_{(2)} \\ &= L(a_{(1)} \otimes S(a_{(2)}))a_{(3)} \\ &= L(a_{(2)} \otimes S(a_{(3)}))a_{(1)}S(a_{(4)})a_{(5)} \\ &= a_{(1)}L(a_{(2)} \otimes S(a_{(3)})) \\ &= (\mathrm{id} \otimes \sigma) \circ \Delta(a). \end{aligned}$$

□

Lemma 5. *The following equations hold:*

- $L^{\star n} \circ (\mathrm{id} \otimes S) \circ \Delta = \sigma^{\star n}$
- $e_\star^{tL} \circ (\mathrm{id} \otimes S) \circ \Delta = e_\star^{t\sigma}$

Proof. We prove this by induction over n. For $n = 0, 1$ the proposition is clear. We calculate

$$\begin{aligned} L^{\star n+1}(a_{(1)} \otimes S(a_{(2)})) &= L \star L^{\star n}(a_{(1)} \otimes S(a_{(2)})) \\ &= L(a_{(1)} \otimes S(a_{(4)}))L^{\star n}(a_{(2)} \otimes S(a_{(3)})) \\ &= L(a_{(1)} \otimes S(a_{(3)}))\sigma^{\star n}(a_{(2)}) \\ &= L(a_{(1)} \otimes S(a_{(2n+2)}))L(a_{(2)} \otimes S(a_{(3)})) \ldots L(a_{(2n)} \otimes S(a_{(2n+1)})) \\ &= L(a_{(1)}\sigma^{\star n}(a_{(2)}) \otimes a_{(3)}) \\ &= L(\sigma^{\star n}(a_{(1)})a_{(2)} \otimes a_{(3)}) \\ &= \sigma^{\star n}(a_{(1)})\sigma(a_{(2)}) \\ &= \sigma^{\star n+1}(a) \end{aligned}$$

The second equation follows easily now:

$$e_\star^{tL} \circ (\mathrm{id} \otimes S) \circ \Delta = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^{\star n} \circ (\mathrm{id} \otimes S) \circ \Delta = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sigma^{\star n} = e_\star^{t\sigma}.$$

□

Theorem 6. *Let $\mathcal{B}$ be a Hopf algebra and L generator of an additive deformation. Then it is a Hopf deformation and the deformed antipodes are given by*

$$S_t = S \star e_\star^{-t\sigma}.$$

Proof. We have to check (7), so we calculate

$$\begin{aligned} \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta(a) &= e_\star^{tL} \star \mu(a_{(1)} \otimes S(a_{(2)}))e_\star^{-t\sigma}(a_{(3)}) \\ &= e_\star^{tL}(a_{(1)} \otimes S(a_{(4)}))a_{(2)}S(a_{(3)})e_\star^{-t\sigma}(a_{(5)}) \\ &= e_\star^{tL}(a_{(1)} \otimes S(a_{(2)}))e_\star^{-t\sigma}(a_{(3)})\mathbb{1} \\ &= e_\star^{t\sigma}(a_{(1)})e_\star^{-t\sigma}(a_{(2)})\mathbb{1} \\ &= \delta(a)\mathbb{1}. \end{aligned}$$

The second equality in (7) follows in the same manner. □

4.4 Constant antipodes in the cocommutative case

Lemma 6. *We have*

$$\partial\sigma = L + L\circ(S\otimes S)\circ\tau.$$

Proof.

$$\begin{aligned}\partial\sigma(a\otimes b) &= \delta(a)\sigma(b) - \sigma(ab) + \sigma(a)\delta(b)\\ &= \delta(a)L(S(b_{(1)})\otimes b_{(2)}) - L(S(a_{(1)}b_{(1)})\otimes a_{(2)}b_{(2)})\\ &\quad + L(S(a_{(1)})\otimes a_{(2)})\delta(b)\\ &= \delta(a)L(S(b_{(1)})\otimes b_{(2)}) - L(S(b_{(1)})S(a_{(1)})\otimes a_{(2)}b_{(2)})\\ &\quad + L(S(a_{(1)})\otimes a_{(2)})\delta(b)\\ &= L(S(b)\otimes S(a)) - L(S(a_{(1)})\otimes a_{(2)}b) + \delta(b)L(S(a_{(1)})\otimes a_{(2)})\\ &= L(S(b)\otimes S(a)) + L(a\otimes b),\end{aligned}$$

where in the fourth equality we used

$$\begin{aligned}0 &= \partial L(S(b_{(1)})\otimes S(a_{(1)})\otimes a_{(2)}b_{(2)})\\ &= \delta(b_{(1)})L(S(a_{(1)})\otimes a_{(2)}b_{(2)}) - L(S(b_{(1)})S(a_{(1)})\otimes a_{(2)}b_{(2)})\\ &\quad + L(S(b_{(1)})\otimes S(a_{(1)})a_{(2)}b_{(2)}) - L(S(b_{(1)})\otimes S(a_{(1)}))\delta(a_{(2)}b_{(2)})\\ &= L(S(a_{(1)})\otimes a_{(2)}b) - L(S(b_{(1)})S(a_{(1)})\otimes a_{(2)}b_{(2)})\\ &\quad + \delta(a)L(S(b_{(1)})\otimes b_{(2)}) - L(S(b)\otimes S(a))\end{aligned}$$

and in the last equality

$$\begin{aligned}0 &= \partial L(S(a_{(1)})\otimes a_{(2)}\otimes b)\\ &= \delta(a_{(1)})L(a_{(2)}\otimes b) - L(S(a_{(1)})a_{(2)}\otimes b)\\ &\quad + L(S(a_{(1)})\otimes a_{(2)}b) - L(S(a_{(1)})\otimes a_{(2)})\delta(b)\\ &= L(a\otimes b) - \delta(a)L(\mathbb{1}\otimes b)\\ &\quad + L(S(a_{(1)})\otimes a_{(2)}b) - L(S(a_{(1)})\otimes a_{(2)})\delta(b).\end{aligned}$$

□

Theorem 7. *Let $\mathcal{B}$ be a Hopf algebra, L generator of an additive deformation. If $\sigma = \sigma\circ S$, then*

$$\widetilde{L} = L - \frac{1}{2}\partial\sigma$$

is the generator of a Hopf deformation with constant antipodes, i.e.

$$\widetilde{\mu}_t\circ(S\otimes\mathrm{id})\circ\Delta = \mathbb{1}\delta = \widetilde{\mu}_t\circ(\mathrm{id}\otimes S)\circ\Delta.$$

Proof. We can write

$$L = \underbrace{\frac{1}{2}(L + L\circ(S\otimes S)\circ\tau)}_{:=L_1} + \underbrace{\frac{1}{2}(L - L\circ(S\otimes S)\circ\tau)}_{:=L_2}$$

Then we have $L_1 = \partial\frac{\sigma}{2}$ and $\sigma_2 = L_2\circ(S\otimes\mathrm{id})\circ\Delta = 0$, since

$$\begin{aligned}L\circ(S\otimes S)\circ\tau\circ(S\otimes\mathrm{id})\circ\Delta &= L\circ(\mathrm{id}\otimes S)\circ(S\otimes S)\tau\circ\Delta\\ &= L\circ(S\otimes\mathrm{id})\circ\Delta\circ S\\ &= \sigma\circ S = \sigma.\end{aligned}$$

□

Lemma 7. *If $\mathcal{B}$ is cocommutative, we have*

$$\sigma = \sigma \circ S.$$

Proof. We calculate

$$\begin{aligned}\sigma \circ S &= -L \circ (S \otimes \mathrm{id}) \circ \Delta \circ S \\ &= -L \circ (S \otimes \mathrm{id}) \circ (S \otimes S) \circ \Delta^{op} \\ &= -L \circ (S^2 \otimes S) \circ \Delta^{op} \\ &= -L \circ (\mathrm{id} \otimes S) \circ \Delta \\ &= \sigma.\end{aligned}$$

□

So when deforming a cocommutative Hopf algebra one can always find an equivalent deformation, such that $S_t = S$ for all $t \in \mathbb{R}$.

5 Examples

Example 1. In this example we realize the algebra of the quantum harmonic oscillator as the essentially only nontrivial additive deformation of the $*$-Hopf algebra of polynomials in adjoint commuting variables $\mathbb{C}[x, x^*]$ with comultiplication and counit defined via

$$\Delta(x^\epsilon) = x^\epsilon \otimes \mathbb{1} + \mathbb{1} \otimes x^\epsilon \qquad \text{and} \qquad \delta(x^\epsilon) = 0,$$

where $\epsilon \in \{1, *\}$.

Proposition. Let $\mathcal{L}$ be an abelian Lie algebra, i.e. $[a, b] = 0\ \forall a, b \in \mathcal{L}$ and consider the universal enveloping Hopf algebra $U(\mathcal{L})$. In the case where $\mathcal{L}$ is of finite dimension n this is just the polynomial algebra in n commuting indeterminates. For two additive deformations $\mu_t^{(1)}, \mu_t^{(2)}$ of $U(\mathcal{L})$ with generators L_1, L_2 the following statements are equivalent:

1. $L_1 - L_2$ is a coboundary i.e. the two deformations differ by a trivial deformation
2. $\mu_t^{(1)}(a \otimes b - b \otimes a) = \mu_t^{(2)}(a \otimes b - b \otimes a)$ for all $a, b \in \mathcal{L}, t \in \mathbb{R}$
3. $L_1(a \otimes b - b \otimes a) = L_2(a \otimes b - b \otimes a)$ for all $a, b \in \mathcal{L}, t \in \mathbb{R}$

Proof. For any additive deformation of $U(\mathcal{L})$ we have

$$\begin{aligned}\mu_t(a \otimes b) &= \mu \star e_\star^{tL}(a \otimes b) \\ &= \mu \otimes e_\star^{tL}(a \otimes b \otimes \mathbb{1} \otimes \mathbb{1} + a \otimes \mathbb{1} \otimes \mathbb{1} \otimes b + \mathbb{1} \otimes b \otimes a \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes a \otimes b) \\ &= ab + tL(a \otimes b)\mathbb{1}\end{aligned}$$

as L is normalized. From this the equivalence of 2 and 3 follows directly and to show that 1 is equivalent to 3 it suffices to show that L is a coboundary iff $L(a \otimes b - b \otimes a) = 0$ for all $a, b \in \mathcal{L}$, since we set $L = L_1 - L_2$.

So let L be a coboundary, i.e. $L = \partial\psi$. It follows that

$$L(a \otimes b - b \otimes a) = -\psi(ab - ba) = 0,$$

since $\mathcal{L}$ is abelian.

Now let $L(a \otimes b - b \otimes a) = 0$ for all $a, b \in \mathcal{L}$. Choose a basis of $\mathcal{L}$ and introduce any ordering on this bases. Then expressions of the form $a_1 \ldots a_n$ with $a_1 \leq \cdots \leq a_n$ form a basis of $U(\mathcal{L})$ define

$$\psi(a_1 \ldots a_n) := \begin{cases} L(a_1 \ldots a_{n-1} \otimes a_n) & \text{with } a_1 \leq \cdots \leq a_n \text{ if } n \geq 2 \\ 0 & \text{else.} \end{cases}$$

We write $\widetilde{L} = L + \partial\psi$ and $\widetilde{\mu}_t = \mu \star e_\star^{t\widetilde{L}}$. Now an easy induction on n shows that $\widetilde{\mu}_t^{(n)}(a_1 \dots a_n) = a_1 \dots a_n$ for $a_1 \leq \dots \leq a_n$. But from the equivalence of 2 and 3 we know that μ_t is commutative so we get $\mu_t = \mu$ for all $t \in \mathbb{R}$. So $\widetilde{L} = L + \partial\psi = 0$ and L is a coboundary. $\square$

It follows that a deformation of $\mathbb{C}[x, x^*]$ is determined up to a trivial deformation by the value of $L(x \otimes x^* - x^* \otimes y) = \mu_1(x \otimes x^* - x^* \otimes x)$. In case of a $*$-deformation L must be hermitian, so this is a real number. Choosing different constants here corresponds to a rescaling of the deformation parameter t so we assume $L(x \otimes x^* - x^* \otimes y) = 1$. There is also a canonical representative for the cohomology class of the generator for which the antipodes are constant. Choosing $L(x \otimes x^*) = -L(x^* \otimes x) = \frac{1}{2}$ one gets $\sigma = 0$.

One gets a well defined $*$-algebra isomorphism from the algebra generated by $a, a^\dagger$ and $\mathbb{1}$ with the relation $aa^\dagger - a^\dagger a = \mathbb{1}$ to the deformation of the polynomial algebra $(\mathbb{C}[x, x^*], \mu_1)$ by setting $\Phi(a) = x$ and $\Phi(a^\dagger) = x^*$. In this sense the quantum harmonic oscillator algebra is the only nontrivial additive deformation of the polynomial algebra in two commuting adjoint variables.

In the last three examples we take as Hopf algebra the group algebra $\mathbb{C}G$ over a group G. We identify linear functionals on $\mathbb{C}G^k$ with functions on G^k for $k \in \mathbb{N}$. For grouplike $a, b \in \mathcal{B}$ we have

$$\mu_t(a \otimes b) = e^{tL(a \otimes b)} ab.$$

Example 2. We saw that in the cocommutative case it is possible to split an additive deformation into a trivial part and a part that corresponds to constant antipodes. But it is still possible that the part with constant antipodes is trivial as this example shows. Consider the 2-coboundary defined by

$$L(m, n) = m^2 n + mn^2$$

on the group algebra of $\mathbb{Z}$. In the following group elements of $\mathbb{Z}$ are denoted (k) to avoid confusion with the complex number k. This is a coboundary, since $L = \partial\psi$ where

$$\psi(k) = -\frac{1}{3}k^3$$

We also see that $L(0, 0) = 0$ and L is commuting, so $L \in B^{(\mathbf{NC})}$. Therefore it generates a trivial deformation. The deformation is nonconstant, since

$$\mu_t((1) \otimes (1)) = e^{L((1),(1))}(2) = 2(2) \neq (2) = \mu((1) \otimes (1)).$$

We calculate

$$\sigma(k) = L((k), (-k)) = -k^3 + k^3 = 0$$

for all $k \in \mathbb{Z}$, so the antipodes are constant. Since the deformation is trivial we can calculate the Φ_t.

$$\Phi_t(k) = e^{t\psi(k)} k = e^{tk^3} k$$

The second way for calculating the S_t yields

$$S_t(k) = \Phi_t \circ S \circ \Phi_t(k) = e^{tk^3} \Phi_t(-k) = e^{tk^3} e^{-tk^3}(0) = (0).$$

So in this situation we have $S \circ \Phi_t = \Phi_{-t} \circ S$.

Example 3. On $\mathbb{Z}^d$ every $d \times d$-matrix A with complex entries defines a 2-cocycle L via

$$L(\underline{k}, \underline{l}) := \underline{k} A \underline{l}^t$$

for $\underline{k}, \underline{l} \in \mathbb{Z}^d$, since the functions $((k_1, \dots, k_d), (l_1, \dots, l_d)) \mapsto k_i l_j$ define cocycles for $i, j = 1, \dots, d$, as is easily checked. These cocycles are of course normalized and commuting, so they are generators of additive deformations on a cocommutative Hopf algebra. L is hermitian iff A is hermitian. We want to apply Theorem 7, so we calculate

$$\sigma(\underline{k}) = L(\underline{k}, -\underline{k}) = -\underline{k} A \underline{k}^t$$

and

$$\begin{aligned}\partial\frac{\sigma}{2}(\underline{k},\underline{l}) &= \frac{1}{2}(-\underline{k}A\underline{k}^t + (\underline{k}+\underline{l})A(\underline{k}+\underline{l})^t - \underline{l}A\underline{l}^t) \\ &= \frac{1}{2}(\underline{k}A\underline{l}^t + \underline{l}A\underline{k}^t) \\ &= \underline{k}\frac{A+A^t}{2}\underline{l}^t\end{aligned}$$

which gives

$$\widetilde{L}(\underline{k},\underline{l}) = (L - \frac{1}{2}\partial\sigma)(\underline{k},\underline{l}) = \underline{k}\frac{A-A^t}{2}\underline{l}^t.$$

So every such cocycle is equivalent to one which comes from an antisymmetric matrix.

Example 4. Let G be a group. then $\mathbb{C}G$ can be turned into a $*$-Hopf algebra in a natural way by extending the map $* : g \mapsto g^{-1}$ antilinearly to the whole of $\mathbb{C}G$. On the group elements the involution $*$ coincides with the antipode S. Now let L be a generator of an additive $*$-deformation, i.e. L is a normalized hermitian 2-cocycle. Then

$$\begin{aligned}\partial\frac{\sigma}{2}(g,h) &= (L + L\circ(S\otimes S)\circ\tau)(g,h) \\ &= \frac{1}{2}(L(g,h) + L(h^*,g^*)) \\ &= \frac{1}{2}(L(g,h) + \overline{L(g,h)}) \\ &= \operatorname{Re} L(g,h)\end{aligned}$$

and consequently

$$\begin{aligned}\widetilde{L}(g,h) &= L - \frac{1}{2}\partial\sigma(g,h) \\ &= \operatorname{Im} L(g,h).\end{aligned}$$

So one has to consider only the case where L is purely imaginary on the group elements.

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