Central Limit Theorem for General Universal Products

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Introduction

We discuss noncommutative notions of independence using an algebraic approach, which essentially defines for $d, m \in \mathbb{N}$ the category of (d,m)-algebraic quantum probability spaces $(\mathscr{A}, \varphi)$, notated by $\mathsf{algP}_{\mathsf{d,m}}$ (see Def. 1). Furthermore, this category is equipped with a so-called *universal, unital, associative product*, for short *u.a.u.-product* (see Def. 2) and we obtain a tensor category. This framework has been introduced by Schürmann [1] for a unified approach to cummulants and covers in particular the well known independcies of tensor, boolean, free, monotonic and antimonotonic. Muraki [2] has shown that these are the only so called *normal* u.a.u.-products in $\mathsf{algP}_{1,1}$, i.e. objects in this category are a pair of an algebra $\mathscr{A}$ and $\varphi \in \mathscr{A}^*$. But we can also model independence in quantum probability, if we assume that φ is a d-tuple of linear functionals $\varphi^{(i)}$ on $\mathscr{A}$ and the algebra $\mathscr{A}$ is structured as an m-fold free product of subalgebras $\mathscr{A}^{(i)} \subseteq \mathscr{A}$. There are good reasons, why to investigate such objects, as concepts of bifreenes[3] $(d=1, m=2)$, c-freenes[4] $(d=2, m=1)$ and recently investigated bimonotonic idependence of type II[5],[6] $(d=1, m=2)$ show. Within this concept of (d,m)-independence we want to present a general version of central limit theorem.

Setting the stage

Definition 1 (Category of (d,m)-algebraic quantum probability spaces [1]) Consider a category C whose objects are a triple $(\mathscr{A}, (\mathscr{A}^{(k)})_{k\in[d]}, (\varphi^{(\ell)})_{\ell\in[m]})$ with following properties

i.) $\mathscr{A}$ is an associative algebra
ii.) $\mathscr{A}$ is ismorphic to the free product $\bigsqcup_{k\in[m]} \mathscr{A}^{(k)}$ and $\forall k \in [m]\colon \mathscr{A}^{(k)}$ is a subalgebra of $\mathscr{A}$
iii.) $\forall \ell \in [d]\colon \varphi^{(\ell)}$ is a linear functional on $\mathscr{A}$

Moreover, assume that the morphisms $j \in \mathsf{Morph}_{\mathsf{C}}((\mathscr{B}, \psi), (\mathscr{A}, \varphi))$ of such category fullfill

iv.) $j \in \mathsf{Morph}_{\mathsf{alg}}(\mathscr{B}, \mathscr{A})$
v.) $\forall k \in [m]\colon j(\mathscr{B}^{(k)}) \subseteq \mathscr{A}^{(k)}$
vi.) $\forall \ell \in [d]\colon \varphi^{(\ell)} \circ j = \varphi^{(\ell)}$

Such a category is called category of (d,m)-algebraic quantum probabilty spaces, denoted by $\mathsf{algP}_{\mathsf{d,m}}$.

Definition 2 (u.a.u.-product in $\mathsf{algP}_{\mathsf{d,m}}$ [1]) A universal product in the category $\mathsf{algP}_{\mathsf{d,m}}$ is a bifunctor $\odot$ of the form

$$\odot\colon \begin{cases} \mathsf{Obj}(\mathsf{algP}_{\mathsf{d,m}} \times \mathsf{algP}_{\mathsf{d,m}}) \ni \Big((\mathscr{A}_1, (\mathscr{A}_1^{(i)})_{i\in[m]}, \varphi_1), (\mathscr{A}_2, (\mathscr{A}_2^{(i)})_{i\in[m]}, \varphi_2)\Big) \\ \qquad \mapsto \Big(\mathscr{A}_1 \sqcup \mathscr{A}_2, (\mathscr{A}_1^{(i)} \sqcup \mathscr{A}_2^{(i)})_{i\in[m]}, (\varphi_1 \odot \varphi_2)\Big) \in \mathsf{Obj}(\mathsf{algP}_{\mathsf{d,m}}) \\ \mathsf{Morph}_{\mathsf{algP}_{\mathsf{d,m}} \times \mathsf{algP}_{\mathsf{d,m}}}\Big(((\mathscr{B}_1, \psi_1), (\mathscr{B}_2, \psi_2)), ((\mathscr{A}_1, \varphi_1), (\mathscr{A}_2, \varphi_2))\Big) \ni (j_1, j_2) \\ \qquad \mapsto j_1 \sqcup j_2 \in \mathsf{Morph}_{\mathsf{algP}_{\mathsf{d,m}}}((\mathscr{B}_1 \sqcup \mathscr{B}_2, \psi_1 \odot \psi_2), (\mathscr{A}_1 \sqcup \mathscr{A}_2, \varphi_1 \odot \varphi_2)) \end{cases}$$

A universal product in $\mathsf{algP}_{\mathsf{d,m}}$ is called unital if for any $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{algP}_{\mathsf{d,m}})$ following equation is satisfied

$$\forall i \in [d], \forall j \in [2]\ \colon (\varphi_1 \odot \varphi_2)^{(i)} \circ \iota_j = \varphi_j^{(i)},$$

where $\iota_j\colon \mathscr{A}_j \to \mathscr{A}_1 \sqcup \mathscr{A}_2, j \in [2]$ are the canonical embeddings. Furthermore, a universal product in $\mathsf{algP}_{\mathsf{d,m}}$ is called associative if for all $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{algP}_{\mathsf{d,m}}), i \in [3]$ holds

$$\Big((\varphi_1 \odot \varphi_2) \odot \varphi_3\Big) = \Big(\varphi_1 \odot (\varphi_2 \odot \varphi_3)\Big).$$

A unital, associative, universal product in the category $\mathsf{algP}_{\mathsf{d,m}}$ is abbreviated by u.a.u.-product.

By stripping away the functional φ for an object $(\mathscr{A}, \varphi) \in \mathsf{Obj}(\mathsf{algP}_{\mathsf{d,m}})$ and only demanding the conditions *iv.*) and *v.*) of Definition 1 we can define in a natural manner the category $\mathsf{alg}_{\mathsf{m}}$. Assuming that a u.a.u.-product in the category $\mathsf{algP}_{\mathsf{d,m}}$ is given, this in a canonical way induces a binary operation on the dual space using the above bifunctor and then projecting onto the 2nd component, i.e.

$$\odot\colon \Big((\mathscr{A}_1)^*\Big)^d \times \Big((\mathscr{A}_2)^*\Big)^d \ni (\varphi_1, \varphi_2) \mapsto \pi_2\Big((\mathscr{A}_1, \varphi_1) \odot (\mathscr{A}_2, \varphi_2)\Big) \in \Big((\mathscr{A}_1 \sqcup \mathscr{A}_2)^*\Big)^d,$$

with properties similar from Definition 2. Furthermore, our interest lies in a subcategory of $\mathsf{alg}_{\mathsf{m}}$.

Definition 3 (The category $\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}$) Consider a category C, where for each $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}) \in \mathsf{Obj}(\mathfrak{C})$, it holds that it also an element of $\mathsf{Obj}(\mathsf{alg}_{\mathsf{m}})$ and

$$\forall k \in [m]\ \exists \text{a family of subspaces } \Big((\mathscr{A}^{(k)})^{\langle\alpha\rangle}\Big)_{\alpha\in\mathbb{N}_0}\colon \mathscr{A}^{(k)} = \bigoplus_{\alpha\in\mathbb{N}_0} (\mathscr{A}^{(k)})^{\langle\alpha\rangle}.$$

Moreover, each $j \in \mathsf{Morph}_{\mathfrak{C}}(\mathscr{B}, \mathscr{A})$ is also an element of $\mathsf{Morph}_{\mathsf{alg}_{\mathsf{m}}}(\mathscr{B}, \mathscr{A})$ and is also homogeneous restricted to $\mathscr{B}^{(k)}$ for all $k \in [m]$. We denote such catgory by $\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}$.

Lemma 1 The bifunctor $\odot$ turns the triple $(\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}, \odot, (\{0\} \mapsto 0))$ into a tensor category with inclusions.

Going to commutative bialgebras

There exists a version of noncommutative central limit theorems in the case of bialgebras. Bialgebras can be seen as comonoids in the tensor category $(\mathsf{alg}_{\mathbb{1}}, \otimes)$ of unital algebras. For the case $d = m = 1$ in [7, Thm. 5.2.4] Lachs has constructed a cotensor functor, which maps comonoids in the tensor category $(\mathsf{alg}^{\mathbb{N}_0}, \sqcup)$, i.e. graded dual semigroups, to graded bialgebras. We try to carry over this *Lachs Functor* to the case $d \neq 1$ or $m \neq 1$. Therefore we need one essential Proposition from Schürmann.

Proposition 1 (Characterization of a universal product [1, Prop. 5.1]) Let $d, m \in \mathbb{N}$ and $\odot$ be a u.a.u-product in $\mathsf{algP}_{\mathsf{d,m}}$. Then for any algebras $\mathscr{A}_i \in \mathsf{Obj}(\mathsf{alg}_{\mathsf{m}})$, $i \in [2]$ there exists a unique mapping

$$\sigma^{\odot}_{\mathscr{A}_1.\mathscr{A}_2}\colon (\mathscr{A}_1 \sqcup \mathscr{A}_2)^d \to \mathrm{S}(\mathscr{A}_1)^{\otimes d} \otimes \mathrm{S}(\mathscr{A}_2)^{\otimes d}$$

such that

$$\forall i \in [2], \forall \varphi_i \in \Big((\mathscr{A}_i)^d\Big)^*\colon (\varphi_1 \odot \varphi_2) = (\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)) \circ \sigma^{\odot}_{\mathscr{A}_1.\mathscr{A}_2}.$$

Hereby $\mathrm{S}(\,\cdot\,)$ denotes the symmetric tensor algebra and $\mathcal{S}(\,\cdot\,)$ is the unique homomorphism from its universal property, i.e. $\mathcal{S}(\varphi) \circ \mathrm{i}_{\mathrm{s}} = \varphi$

The Lachs Functor

Proposition 1 enables us to construct for all $d, m \in \mathbb{N}$ a cotensor functor (*Lachs Functor*) between $\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}$ and the category of graded, commutative, unital algebras $\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}$

$$\mathfrak{S}\colon \begin{cases} \mathsf{Obj}(\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}) \ni \Big(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, ((\mathscr{A}^{(i)})^{\langle\alpha\rangle})_{(i,\alpha)\in[m]\times\mathbb{N}_0}\Big) \\ \qquad \mapsto \Big(\mathrm{S}(\mathscr{A}^d), ((\mathrm{S}(\mathscr{A}^d))^{\langle\alpha\rangle})_{\alpha\in\mathbb{N}_0}\Big) \in \mathsf{Obj}(\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}) \\ \mathsf{Morph}_{\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}}(\mathscr{B}, \mathscr{A}) \ni j \mapsto \mathrm{S}(j^d) \in \mathsf{Morph}_{\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}}(\mathrm{S}(\mathscr{B}^d), \mathrm{S}(\mathscr{A}^d)) \end{cases}$$

Basically, we see the only modification of the functor $\mathfrak{S}$ in comparison to the functor S is, that $\mathfrak{S}$ "forgets" about the free generation of an algebra $\mathscr{A}$ and d copies of $\mathscr{A}$ are created. Moreover, the symmetric tensor algebra $\mathrm{S}((\mathscr{A})^d)$ is equipped with a certain $\mathbb{N}_0$-gradation.

Theorem 1

Consider the tensor categories $(\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}, \sqcup, \{0\})$ and $(\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}, \otimes, \mathbb{C})$, then the triple $(\mathfrak{S}, \mathcal{S}(\sigma^{\odot}), g_0)$ consisting of the functor $\mathfrak{S}\colon \mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0} \to \mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}$, the natural transformation $\mathcal{S}(\sigma^{\odot})\colon \mathfrak{S}(\,\cdot\, \sqcup \,\cdot\,) \Rightarrow \mathfrak{S}(\,\cdot\,) \otimes \mathfrak{S}(\,\cdot\,)$ and the map $g_0\colon \mathrm{S}(\{0\}^d) \to \mathbb{C}$ with $g_0(\mathbb{1}_{\mathrm{S}(\{0\}^d)}) = 1$ is a cotensor functor.

The situation resulting from Theorem 1 is illustrated in following diagram.

Tensor category $(\mathsf{alg}_m^{\mathbb{N}_0}, \odot, \{0\})$ —— cotensor functor $(\mathfrak{S}, \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}}), g_0)$ ——→ **Tensor category** $(\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}, \otimes, \mathbb{C})$

comonoid in ↑ | ↑ comonoid in

Dual semigroup $(\mathscr{D}, \Delta, 0)$ ↪—— as subalgebra ——→ **Bialgebra** $(\mathfrak{S}(\mathscr{D}), \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}}) \circ \mathfrak{S}(\Delta), g_0 \circ \mathfrak{S}(0))$

Assume $\odot$ on $\mathsf{alg}_{\mathsf{m}}$ is given and a comonoid $(\mathscr{D}, \Delta, \delta)$ in the category $\mathsf{alg}_{\mathsf{m}}$ is chosen, then we define the *convolution product*

$$\divideontimes\colon \Big(\mathscr{D}^*\Big)^d \times \Big(\mathscr{D}^*\Big)^d \ni (\varphi_1, \varphi_2) \mapsto \Big((\varphi_1 \odot \varphi_2)^{(i)} \circ \Delta\Big)_{i\in[d]} \in \Big(\mathscr{D}^*\Big)^d.$$

Corollary 1 The prescription $(\mathscr{D}^*)^d \ni \varphi \mapsto \mathcal{S}(\varphi) \in ((\mathcal{S}(\mathscr{D}))^*)^d$ defines an homomorphism between the monoids $((\mathscr{D}^*)^d, \divideontimes)$ and $(((\mathcal{S}(\mathscr{D}))^*)^d, \star)$, where $\star$ is the ordinary convolution of coalgebras.

Convolution exponential for $\divideontimes$

If we want to define a convolution exponential for $\divideontimes$, then a naive definition fails, because $\divideontimes$ does not distribute over $+$. Looking at the the characterization of *weakly continuous* semigroups, we can see, what might be a good definition for the convolution exponential of $\divideontimes$. We recall that a family $(\varphi_t)_{t\in\mathbb{R}_+} \subseteq (\mathscr{D}^d)^*$ is a called a convolution semigroup on the comonoid $(\mathscr{D}, \Delta, \delta)$ if for all $s, t \in \mathbb{R}_+$ we have $\varphi_s \divideontimes \varphi_t = \varphi_{s+t}$ and $\varphi_0 = \delta^d \equiv \mathsf{0}$. Then a convolution semigroup is said to be weakly continuous if in addition $\forall b \in \mathscr{D}^d\colon \lim_{t\to 0^+} \varphi_t(b) = \delta^d(b) \equiv 0$.

Proposition 2 (Characterization of convolution semigroup on comonoids in $\mathsf{alg}_{\mathsf{m}}$) Let $(\mathscr{D}, \Delta, \mathsf{0})$ be a comonoid in $\mathsf{alg}_{\mathsf{m}}$ and $(\varphi_t)_{t\in\mathbb{R}_+} \subseteq (\mathscr{D}^d)^*$ be convolution semigroup on $\mathscr{D}$. Following assertions are equivalent

i.) The convolution semigroup $(\varphi_t)_{t\in\mathbb{R}_+}$ is weakly continuous.
ii.) There exists a linear functional $\Psi \in (\mathscr{D}^d)^*$ such that

$$\forall t \in \mathbb{R}_+\colon \varphi_t = \exp_\star(t\,\mathrm{D}(\Psi))\restriction_{\mathscr{D}^d} =: \exp_{\divideontimes} \Psi$$

If one of the above conditions is fullfilled, then the linear functional Ψ is uniquely determined by $(\varphi_t)_{t\in\mathbb{R}_+}$, i.e. $\forall b \in \mathscr{D}^d\colon \Psi(b) = \lim_{t\to 0^+}(\varphi_t(b)/t)$. The map $\mathrm{D}(\Psi)$ is defined on $\mathrm{S}(\mathscr{D}^d) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{S}^n(\mathscr{D}^d)$ by

$$\mathrm{D}(\Psi)\restriction_{\mathrm{S}^n(\mathscr{D}^d)} := \begin{cases} 0, & \text{if } n = 0 \text{ or } n > 1 \\ \Psi, & \text{if } n = 1, \end{cases}$$

Central limit theorem

The first noncommutative version of a central limit theorem dates back to von Waldenfels [8]. We present a version, whereby its proof is reduced to the well known bialgebra case. Let V be an $\mathbb{N}_0$-graded vector space and choose $z \in \mathbb{C}$. We define the linear map $S_z\colon V \ni v \mapsto z^{\deg v} v \in V$ for homogeneous $v \in V$.

Theorem 2

Let $d, m, \nu \in \mathbb{N}$ and $\odot$ be universal product on $\mathsf{alg}_{\mathsf{m}}$ and assume that $\mathscr{D} \in \mathsf{Obj}(\mathsf{comon}(\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}))$ with induced $\mathbb{N}_0$-gradation $(\mathscr{D}^{\langle\alpha\rangle})_{\alpha\in\mathbb{N}_0}$. If $\varphi \in (\mathscr{D}^d)^*$ fullfills $\varphi\restriction_{(\mathscr{D}^d)^{\langle\alpha\rangle}} = 0$ for $0 \le \alpha < \nu$, then

$$\forall b \in \mathscr{D}^d\colon \lim_{n\to\infty} \Big(\varphi^{\divideontimes n} \circ S_{n^{-\frac{1}{\nu}}}\Big)(b) = (\exp_{\divideontimes}(g_\varphi))(b)$$

where $g_\varphi \in (\mathscr{D}^d)^*$ defined by $g_\varphi\restriction_{(\mathscr{D}^d)^{\langle\alpha\rangle}} = \varphi\restriction_{(\mathscr{D}^d)^{\langle\alpha\rangle}}$ if $\alpha = \nu$ and 0 otherwise.

References

[1] Manzel, Sarah; Schürmann, Michael: *Non-commutative stochastic independence and cumulants.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. 20.2 (2017), 1750010, 38.
[2] Muraki, Naofumi: *The five independences as natural products.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6.3 (2003).
[3] Voiculescu, Dan-Virgil: *Free probability for pairs of faces I.* Comm. Math. Phys. 332.3 (2014), 955–980.
[4] Bożejko, Marek; Speicher, Roland: *ψ-independent and symmetrized white noises.* In: Quantum probability & related topics. QP-PQ, VI, 219–236. World Sci. Publ., River Edge, NJ, 1991.
[5] Gerhold, Malte: *Bimonotone Brownian Motion.* Preprint **arXiv:1708.0351v1** (2017), 14 pages.
[6] Gu, Yinzheng; Hasebe,Takahiro; Skoufranis, Paul: *Bi-monotonic independence for pairs of algebras.* Preprint **arXiv:1708.05334v2** (2017), 31 pages.
[7] Lachs, Stephanie: *A New Family of Universal Products and Aspects of a Non-Positive Quantum Probability Theory.* PhD thesis. Ernst-Moritz-Arndt-Universität Greifswald, 2015.
[8] von Waldenfels, Wilhem: *An algebraic central limit theorem in the anti-commuting case.* Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 42 (1978), 135-140.