Central Limit Theorem for General Universal Products

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Séminaire Analyse Fonctionnelle, Laboratoire de Mathématiques de Besançon, 12.12.2017

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Motivation

- investigate noncommutative notions of independence using an algebraic approach
- for $d, m \in \mathbb{N}$ define category $\mathsf{algP_{d,m}}$ of (d, m)-algebraic quantum probability spaces [MS17][1]
- model independence by so called universal products
- Muraki has shown $\exists$ only 5 *normal* universal products in $\mathsf{algP_{1,1}}$, i.e. objects are algebras $\mathscr{A}$ equipped with $\varphi \in \mathscr{A}^*$
- What about a d-tuple of linear functionals? What about an m-fold free product of $\mathscr{A}$?
- reasons why we should study such structures are e.g. bifreeness $(d = 1, m = 2)$ [Voi14], c-freeness $(d = 2, m = 1)$ [BS91] and bimonotonic independence of type II $(d = 1, m = 2)$ [GHS17] [Ger17]

[1] S. Manzel and M. Schürmann. "Non-commutative stochastic independence and cumulants". In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 20.2 (2017), pp. 1750010, 38.

Contents

Notational conventions

- for $m \in \mathbb{N}$ set $[m] := \{1, \dots, m\}$
- set of all linear functionals of vector space V denoted by V^*
- $\mathrm{S}(V)$ for symmetric tensor algebra of $\mathbb{C}$-vector space V
- if $\mathscr{A}$ is unital algebra and $f\colon V \to \mathscr{A}$ is linear map with $f(x)f(y) = f(y)f(x)$ f.a. $x, y \in V$, then $\mathcal{S}(f)\colon \mathrm{S}(V) \to \mathscr{A}$ is unique unital algebra homomorphism s.t. $\mathcal{S}(f) \circ \iota_V = f$.
- let V, W be $\mathbb{C}$-vector spaces and $g\colon V \to W$ linear map, we put $\mathrm{S}(g) := \mathcal{S}(\iota_W \circ g)\colon \mathrm{S}(V) \to \mathrm{S}(W)$
- all algebras of consideration are in particular $\mathbb{C}$-vector spaces, associative but not necessarilly unital
- free product of algebras?

Digression: Free product of algebras

- for arbitrary index set I define
  $\mathbb{A}_I := \{ \varepsilon = (\varepsilon_i)_{i\in[m]} \in I^m \mid m \in \mathbb{N}, \varepsilon_k \neq \varepsilon_{k+1}, k = 1, \dots, m-1 \}$
- given family of vector spaces $(V_i)_{i\in I}$, for $\varepsilon \in \mathbb{A}_I$ set
  $V_\varepsilon := V_{\varepsilon_1} \otimes \cdots \otimes V_{\varepsilon_m}$
- given family of algebras $(\mathscr{A}_i)_{i\in I}$ we set

$$\bigsqcup_{i\in I} \mathscr{A}_i := \bigoplus_{\varepsilon\in\mathbb{A}_I} \mathscr{A}_\varepsilon$$

  with multiplication given by

$$\underbrace{(a_1 \otimes \cdots \otimes a_m)}_{\in\mathscr{A}_\varepsilon}\underbrace{(b_1 \otimes \cdots \otimes b_n)}_{\in\mathscr{A}_\delta} := \begin{cases} a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n & \text{if } \varepsilon_m \neq \delta_1 \\ a_1 \otimes \cdots \otimes a_m b_1 \otimes \cdots \otimes b_n & \text{if } \varepsilon_m = \delta_1 \end{cases}$$

- $\sqcup$ is coproduct in category alg, i.e. for $(\mathscr{A}_i)_{i\in I} \subseteq \mathsf{Obj}(\mathsf{alg})$, $\mathscr{A} \in \mathsf{Obj}(\mathsf{alg})$ and family of morphisms $(f_i \colon \mathscr{A}_i \to \mathscr{A})_{i\in I}$ exists unique morphism $\bigsqcup_{i\in I} f_i \colon \bigsqcup_{i\in I} \mathscr{A}_i \to \mathscr{A}$ such that $\bigsqcup_{i\in I} f_i \circ \iota_i = f_i$. Moreover

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\iota_i"'] & Y_i \arrow[d, "\iota_i'"] \\ \bigsqcup_{i\in I} X_i \arrow[r, "\bigsqcup_{i\in I} f_i"'] & \bigsqcup_{i\in I} Y_i \end{tikzcd}$$

Central Limit Theorem for General Universal Products

Definition 1 (Category $\mathsf{alg_m}$)

- objects of category $\mathsf{alg_m}$ are ordered pairs $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[d]})$ with following properties
  - *i.)* $\mathscr{A}$ is an associative algebra
  - *ii.)* $\forall\, i \in [m] \colon \mathscr{A}^{(i)}$ is a subalgebra of $\mathscr{A}$
  - *iii.)* finite family $(\mathscr{A}^{(i)})_{i\in[m]}$ *freely generates* $\mathscr{A}$, i.e. the algebra homomorphism from $\bigsqcup_{i\in[m]} \mathscr{A}^{(i)} \to \mathscr{A}$ defined by

    $$\mathscr{A}_\varepsilon \ni a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdot \cdots \cdot a_n \in \mathscr{A}$$

    is bijection
- morphisms $j \in \mathsf{Morph}_{\mathsf{alg_m}}\big((\mathscr{B}, (\mathscr{B}^{(i)})_{i\in[m]}), (\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]})\big)$ fullfill the following
  - *iv.)* $j \in \mathsf{Morph}_{\mathsf{alg}}(\mathscr{B}, \mathscr{A})$
  - *v.)* $\forall\, k \in [m] \colon j(\mathscr{B}^{(k)}) \subseteq \mathscr{A}^{(k)}$

Definition 2 (Category $\mathsf{algP_{d,m}}$ [MS17, Sec. 2])

- objects of $\mathsf{algP_{d,m}}$ are triples $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, (\varphi^{(i)})_{i\in[d]})$, wherein $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}) \in \mathsf{Obj}(\mathsf{alg_m})$ and $(\varphi^{(i)})_{i\in[d]} \in (\mathscr{A}^*)^d$
- for morphisms $j \in \mathsf{Morph}_{\mathsf{algP_{d,m}}}\big((\mathscr{B}, (\mathscr{B}^{(i)})_{i\in[m]}, (\psi^{(i)})_{i\in[d]}), (\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, (\varphi^{(i)})_{i\in[d]})\big)$ of $\mathsf{algP_{d,m}}$ we demand
  - *i.)* $j \in \mathsf{Morph}_{\mathsf{alg_m}}(\mathscr{B}, \mathscr{A})$
  - *ii.)* $\forall i \in [d]\colon \varphi^{(i)} \circ j = \psi^{(i)}$.

Remark 1 there exist well known isomorphisms

$$\mathsf{Hom}_{\mathbb{C}}(V^d, \mathbb{C}) \cong \big(\mathsf{Hom}_{\mathbb{C}}(V, \mathbb{C})\big)^d \cong \mathsf{Hom}_{\mathbb{C}}(V, \mathbb{C}^d)$$

Definition 3 (u.a.u.-product in $\mathsf{algP_{d,m}}$)

- universal product in the category $\mathsf{algP_{d,m}}$ is bifunctor $\odot$ of the form

$$\odot : \begin{cases} \mathsf{Obj}(\mathsf{algP_{d,m}} \times \mathsf{algP_{d,m}}) \ni \Big(\big(\mathscr{A}_1, (\mathscr{A}_1^{(i)})_{i\in[m]}, \varphi_1\big), (\mathscr{A}_2, (\mathscr{A}_2^{(i)})_{i\in[m]}, \varphi_2 \Big) \\ \qquad \mapsto \Big(\mathscr{A}_1 \sqcup \mathscr{A}_2, (\mathscr{A}_1^{(i)} \sqcup \mathscr{A}_2^{(i)})_{i\in[m]}, \varphi_1 \odot \varphi_2\Big) \in \mathsf{Obj}(\mathsf{algP_{d,m}}) \\ \mathsf{Morph}_{\mathsf{algP_{d,m}} \times \mathsf{algP_{d,m}}}\Big(\big((\mathscr{B}_1, \psi_1), (\mathscr{B}_2, \psi_2)\big)\big), \big((\mathscr{A}_1, \varphi_1), (\mathscr{A}_2, \varphi_2)\big)\Big) \ni (j_1, j_2) \\ \qquad \mapsto j_1 \sqcup j_2 \in \mathsf{Morph}_{\mathsf{algP_{d,m}}}\big((\mathscr{B}_1 \sqcup \mathscr{B}_2, \psi_1 \odot \psi_2), (\mathscr{A}_1 \sqcup \mathscr{A}_2, \varphi_1 \odot \varphi_2)\big) \end{cases}$$

- universal product in $\mathsf{algP_{d,m}}$ is called unital if

$$\forall i \in [d], \forall j \in [2] \; : (\varphi_1 \odot \varphi_2)^{(i)} \circ \iota_j = \varphi_j^{(i)}$$

- universal product in $\mathsf{algP_{d,m}}$ is called associative if

$$\Big((\varphi_1 \odot \varphi_2) \odot \varphi_3\Big) = \Big(\varphi_1 \odot (\varphi_2 \odot \varphi_3)\Big).$$

product having all these 3 properties is abbreviated by u.a.u.-product

- tensor category is category $\mathfrak{C}$ with bifunctor $\boxtimes\colon \mathfrak{C}\times\mathfrak{C}\to\mathfrak{C}$, that is associative up to natural isomorphism, and an object E that is both a left and right identity for $\boxtimes$. coherence conditions ensure that all relevant diagrams commute

Remark 2

i.) if $\odot$ is an u.a.u.-product in $\mathsf{algP_{d,m}}$ $\Rightarrow$ universality condition, i.e. for all $\mathscr{B}_i \in \mathsf{Obj}(\mathsf{alg_m})$ and $(\mathscr{A}_i, \varphi_i) \in \mathsf{Obj}(\mathsf{algP_{d,m}})$ and $j_i \in \mathsf{Morph}_{\mathsf{alg_m}}(\mathscr{B}_i, \mathscr{A}_i), i \in [2]$ holds

$$\forall \ell \in [d]\colon \Big((\varphi_1^{(i)} \circ j_1)_{i\in[d]} \odot (\varphi_2^{(i)} \circ j_2)_{i\in[d]}\Big)^{(\ell)} = (\varphi_1 \odot \varphi_2)^{(\ell)} \circ (j_1 \sqcup j_2)$$

ii.) if $\odot$ is an u.a.u.-product in $\mathsf{algP_{d,m}}$, then $(\mathsf{algP_{d,m}}, \odot, (\{0\}, 0\mapsto 0))$ is tensor category

Definition 4 if $\odot$ in $\mathsf{algP_{d,m}}$ and $(\mathscr{A}_i, (\mathscr{A}_i^{(k)})_{k\in[d]}) \in \mathsf{Obj}(\mathsf{alg_m}), i \in [2]$ are given, then define for $\varphi_i \in ((\mathscr{A}_i)^*)^d$

$$\varphi_1 \odot \varphi_2 := \Big(\big(\mathscr{A}_1, (\mathscr{A}_1^{(k)})_{k\in[d]}, \varphi_1\big) \odot \big(\mathscr{A}_2, (\mathscr{A}_2^{(k)})_{k\in[d]}, \varphi_2\big)\Big)$$

where we identify $\varphi_1 \odot \varphi_2 \in ((\mathscr{A}_1 \sqcup \mathscr{A}_2)^*)^d$

Definition 5 (Catgegory $\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}$ and $\mathsf{calg}_\mathbb{1}^{\mathbb{N}_0}$)

i.)
- objects of $\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}$ are triples $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, ((\mathscr{A}^{(i)})^{\langle\alpha\rangle})_{(i,\alpha)\in[m]\times\mathbb{N}_0})$ where $(\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}) \in \mathsf{Obj}(\mathsf{alg}_\mathsf{m})$ and $\forall i \in [m]\ \exists$ a family of subspaces $\big((\mathscr{A}^{(i)})^{\langle\alpha\rangle}\big)_{\alpha\in\mathbb{N}_0}$, such that $\mathscr{A}^{(i)} = \bigoplus_{\alpha\in\mathbb{N}_0}\big(\mathscr{A}^{(i)}\big)^{\langle\alpha\rangle}$ and $\mathscr{A}^{(i)}$ is a graded algebra
- morphisms $j \in \mathsf{Morph}_{\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}}(\mathscr{B}, \mathscr{A})$ are elements of $\mathsf{Morph}_{\mathsf{alg}_\mathsf{m}}(\mathscr{B}, \mathscr{A})$ and $j\restriction_{\mathscr{B}^{(i)}}, i \in [m]$ is also homogeneous

ii.)
- objects of $\mathsf{calg}_\mathbb{1}^{\mathbb{N}_0}$ are commutative, unital, $\mathbb{N}_0$-graded algebras
- morphisms are homogeneous, unital algebra homomorphisms

Lemma 1 $(\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}, \sqcup, \{0\})$ and $(\mathsf{calg}_\mathbb{1}^{\mathbb{N}_0}, \otimes, \mathbb{C})$ are tensor categories

Proposition 1 ([MS17, Prop. 5.1]) if $\odot$ is u.a.u.-product in $\mathsf{algP_{d,m}}$
$\Rightarrow \forall \mathscr{A}_i \in \mathsf{Obj}(\mathsf{alg_m})$, $i \in [2]$ $\exists!$ linear mapping

$$\sigma^{\odot}_{\mathscr{A}_1 . \mathscr{A}_2} : (\mathscr{A}_1 \sqcup \mathscr{A}_2)^d \to \mathrm{S}(\mathscr{A}_1)^{\otimes d} \otimes \mathrm{S}(\mathscr{A}_2)^{\otimes d}$$

such that for all $\varphi_i \in \left((\mathscr{A}_i)^d\right)^*, i \in [d]$ the diagram is commutative

$$\begin{tikzcd}
(\mathscr{A}_1 \sqcup \mathscr{A}_2)^d \arrow[rr, "\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}"] \arrow[dr, "\varphi_1 \odot \varphi_2"'] & & \mathrm{S}(\mathscr{A}_1)^{\otimes d} \otimes \mathrm{S}(\mathscr{A}_2)^{\otimes d} \arrow[dl, "\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)"] \\
& \mathbb{C} &
\end{tikzcd}.$$

Lemma 2 stated linear mappings $\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}$ are even homogeneous

Definition 6 (Lachs Functor, [Lac15][2])

$$\mathfrak{S}\colon \begin{cases} \mathsf{Obj}(\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}) \ni (\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, ((\mathscr{A}^{(i)})^{\langle\alpha\rangle})_{(i,\alpha)\in([m]\times\mathbb{N}_0)}) \\ \qquad \mapsto \Big(\mathrm{S}(\mathscr{A}^d), \big((\mathrm{S}(\mathscr{A}^d))^{\langle\alpha\rangle}\big)_{\alpha\in\mathbb{N}_0}\Big) \in \mathsf{Obj}(\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}) \\ \mathsf{Morph}_{\mathsf{alg}_{\mathsf{m}}^{\mathbb{N}_0}}(\mathscr{B}, \mathscr{A}) \ni j \mapsto \mathrm{S}(j^d) \in \mathsf{Morph}_{\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}}(\mathrm{S}(\mathscr{B}^d), \mathrm{S}(\mathscr{A}^d)) \end{cases}$$

Theorem 1 ([Lac15, Thm. 5.2.4]) if $\odot$ u.a.u.-product, $g_0\colon \mathrm{S}(\{0\}^d) \to \mathbb{C}$ with $g_0(\mathbb{1}_{\mathrm{S}(\{0\}^d)}) = 1$, then $(\mathfrak{S}, \mathcal{S}(\sigma^{\odot}), g_0)$ is cotensor functor

- What does this mean???
- What are the consequences???

[2] S. Lachs. "A New Family of Universal Products and Aspects of a Non-Positive Quantum Probability Theory". PhD thesis. Ernst-Moritz-Arndt-Universität Greifswald, 2015.

Definition 7 (comonoid) if $(\mathfrak{C}, \boxtimes)$ is tensor category, a comonoid $(\mathscr{C}, \Delta, \delta)$ is object with morphisms

- $\Delta\colon \mathscr{C} \to \mathscr{C} \boxtimes \mathscr{C}$ (comultiplication)
- $\delta\colon \mathscr{C} \to E$ (counit)

such that following diagrams commute

$$\begin{tikzcd}
\mathscr{C} \boxtimes \mathscr{C} \arrow[d, "\mathsf{id}_{\mathscr{C}} \boxtimes \Delta"'] & \mathscr{C} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & \mathscr{C} \boxtimes \mathscr{C} \arrow[d, "\Delta \boxtimes \mathsf{id}_{\mathscr{C}}"] \\
\mathscr{C} \boxtimes (\mathscr{C} \boxtimes \mathscr{C}) \arrow[rr, "\alpha_{\mathscr{C},\mathscr{C},\mathscr{C}}"'] & & (\mathscr{C} \boxtimes \mathscr{C}) \boxtimes \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
E \boxtimes \mathscr{C} \arrow[dr, "\ell_{\mathscr{C}}"'] & \mathscr{C} \boxtimes \mathscr{C} \arrow[l, "\delta \boxtimes \mathsf{id}_{\mathscr{C}}"'] \arrow[r, "\mathsf{id}_{\mathscr{C}} \boxtimes \delta"] & \mathscr{C} \boxtimes E \arrow[dl, "r_{\mathscr{C}}"] \\
& \mathscr{C} \arrow[u, "\Delta"'] &
\end{tikzcd}$$

Example 1 bialgebra is comonoid in $(\mathsf{alg}_{\mathbb{1}}, \otimes)$ and dual semigroup is comonoid in $(\mathsf{alg}_{\mathsf{m}}, \sqcup)$

Definition 8 (cotensor functor) given $(\mathfrak{C}, \boxtimes)$,$(\mathfrak{C}', \boxtimes')$ tensor categories with unit objects E and E', then cotensor functor is triple $(\mathsf{F}, \mathcal{T}, g_0)$, where

i.) $\mathsf{F} \colon \mathfrak{C} \to \mathfrak{C}'$ is functor

ii.) $\mathcal{T} \colon \mathsf{F}(\,\cdot \boxtimes \cdot\,) \Rightarrow \mathsf{F}(\,\cdot\,) \boxtimes' \mathsf{F}(\,\cdot\,)$ is natural transformation

iii.) a morphism $g_0 \colon E \to E'$

such that coassociativity and counit diagrams commute *(not given here)*

$$\begin{tikzcd}
\mathsf{F}(A \boxtimes (B \boxtimes C)) \arrow[r, "\mathsf{F}(\alpha_{A,B,C})"] \arrow[d, "\mathcal{T}_{A,B\boxtimes C}"'] & \mathsf{F}((A \boxtimes B) \boxtimes C) \arrow[d, "\mathcal{T}_{A\boxtimes B,C}"] \\
\mathsf{F}(A \boxtimes B) \boxtimes' \mathsf{F}(C) \arrow[d, "\mathrm{id}_{\mathsf{F}(A)} \boxtimes' \mathcal{T}_{B,C}"'] & \mathsf{F}(A) \boxtimes' \mathsf{F}(B \boxtimes C) \arrow[d, "\mathcal{T}_{A,B} \boxtimes' \mathrm{id}_{\mathsf{F}(C)}"] \\
\mathsf{F}(A) \boxtimes' (\mathsf{F}(B) \boxtimes' \mathsf{F}(C)) \arrow[r, "\alpha'_{\mathsf{F}(A),\mathsf{F}(B),\mathsf{F}(C)}"'] & (\mathsf{F}(A) \boxtimes' \mathsf{F}(B)) \boxtimes' \mathsf{F}(C)
\end{tikzcd}$$

Theorem 2 (Cotensor functor preserves comonoids [Lac15, Cor. 2.3.5]) for cotensor functor $(\mathsf{F}, \mathcal{T}, g_0) \colon (\mathfrak{C}, \boxtimes) \to (\mathfrak{C}', \boxtimes')$ and any comonoid $(\mathscr{C}, \Delta, \delta)$ in $(\mathfrak{C}, \boxtimes)$, the triple $(\mathsf{F}(\mathscr{C}), \mathcal{T}_{\mathscr{C},\mathscr{C}} \circ \Delta, g_0 \circ \mathsf{F}(\delta))$ is comonoid in $(\mathfrak{C}', \boxtimes')$

$$\mathfrak{S}\colon \begin{cases} \mathsf{Obj}(\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}) \ni (\mathscr{A}, (\mathscr{A}^{(i)})_{i\in[m]}, ((\mathscr{A}^{(i)})^{\langle\alpha\rangle})_{(i,\alpha)\in([m]\times\mathbb{N}_0)}) \\ \qquad \mapsto \Big(\mathrm{S}(\mathscr{A}^d), \big((\mathrm{S}(\mathscr{A}^d))^{\langle\alpha\rangle}\big)_{\alpha\in\mathbb{N}_0}\Big) \in \mathsf{Obj}(\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}) \\ \mathsf{Morph}_{\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}}(\mathscr{B}, \mathscr{A}) \ni j \mapsto \mathrm{S}(j^d) \in \mathsf{Morph}_{\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}}(\mathrm{S}(\mathscr{B}^d), \mathrm{S}(\mathscr{A}^d)) \end{cases}$$

- apply scenario to $\mathfrak{S}(\,\cdot\sqcup\cdot\,) := \mathfrak{S}\circ\sqcup\colon \mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}\times\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0} \to \mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}$ and $\mathfrak{S}(\,\cdot\,)\otimes\mathfrak{S}(\,\cdot\,) := \otimes\circ(\mathfrak{S},\mathfrak{S})\colon \mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}\times\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0} \to \mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}$
- for $(\mathcal{S}(\sigma^{\odot}_{\mathscr{A}_1,\mathscr{A}_2}))_{\mathscr{A}_1,\mathscr{A}_2\in\mathsf{Obj}(\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0})} =: \mathcal{S}(\sigma_{\cdot,\odot})$, we have $\mathcal{S}(\sigma^{\odot})\colon\ \mathfrak{S}(\,\cdot\sqcup\cdot\,) \Rightarrow \mathfrak{S}(\,\cdot\,)\otimes\mathfrak{S}(\,\cdot\,)$

Theorem 3 ([Lac15, Thm. 5.2.4]) put $g_0\colon \mathrm{S}(\{0\}^d)\to\mathbb{C}$ with $g_0(\mathbb{1}_{\mathrm{S}(\{0\}^d)}) = 1$, then

$$\begin{tikzcd}[column sep=huge] \substack{\textbf{Tensor category} \\ (\mathsf{alg}_m^{\mathbb{N}_0}, \odot, \{0\})} \arrow[r, "\text{cotensor functor}", "{(\mathfrak{S}, \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}}), g_0)}"'] & \substack{\textbf{Tensor category} \\ (\mathsf{calg}_{\mathbb{1}}^{\mathbb{N}_0}, \otimes, \mathbb{C})} \\ \substack{\textbf{Dual semigroup} \\ (\mathscr{D}, \Delta, 0)} \arrow[u, dashed, "\text{comonoid in}"] \arrow[r, hook, "\text{as subalgebra}"] & \substack{\textbf{Bialgebra} \\ (\mathfrak{S}(\mathscr{D}), \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}})\circ\mathfrak{S}(\Delta), g_0\circ\mathfrak{S}(0))} \arrow[u, dashed, "\text{comonoid in}"'] \end{tikzcd}$$

Central Limit Theorem for General Universal Products

Recall convolution product for comonoid $(\mathscr{B}, \Delta, \delta)$ in $\mathsf{alg}_{\mathbb{1}}$ (bialgebras)

$$\star\colon \mathscr{B}^* \times \mathscr{B}^* \ni (\varphi_1, \varphi_2) \mapsto (\varphi_1 \otimes \varphi_2) \circ \Delta \in \mathscr{B}^*$$

Definition 9 (Convolution product for $\odot$) given u.a.u.-product in $\mathsf{algP}_{\mathsf{d},\mathsf{m}}$ define convolution product for comonoid in $(\mathscr{D}, \Delta, \mathsf{0})$ in $(\mathsf{alg}_{\mathsf{m}}, \sqcup)$ by

$$\divideontimes\colon \left(\mathscr{D}^*\right)^d \times \left(\mathscr{D}^*\right)^d \ni (\varphi_1, \varphi_2) \mapsto \left((\varphi_1 \odot \varphi_2)^{(i)} \circ \Delta\right)_{i\in[d]} \in \left(\mathscr{D}^*\right)^d$$

Lemma 3 prescription $(\mathscr{D}^d)^* \ni \varphi \mapsto \mathcal{S}(\varphi) \in \left(\mathrm{S}(\mathscr{D}^d)\right)^*$ is homomorphism between monoids $((\mathscr{D}^d)^*, \divideontimes)$ and $((\mathrm{S}(\mathscr{D}^d))^*, \star)$, i.e. for $\varphi_i \in (\mathscr{D}^d)^*$, $i \in [2]$ holds

$$\mathcal{S}(\varphi_1 \divideontimes \varphi_2) = \mathcal{S}(\varphi_1) \star \mathcal{S}(\varphi_2).$$

Idea of Proof:

$$
\begin{tikzcd}
\mathscr{D}^d \arrow[d, "\iota_{\mathrm{s}}"'] \arrow[dr, "\varphi_1 \circledast \varphi_2"] & \\
\mathrm{S}(\mathscr{D}^d) \arrow[r, "\mathcal{S}(\varphi_1) \star \mathcal{S}(\varphi_2)"'] & \mathbb{C}
\end{tikzcd}
$$

direct computation shows

$$
\begin{aligned}
\Big(\mathcal{S}(\varphi_1) \star \mathcal{S}(\varphi_2)\Big) \circ \iota_{\mathrm{s}} &\overset{(1)}{=} \bigg(\Big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\Big) \circ \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}}) \circ \mathrm{S}(\Delta^d)\bigg) \circ \iota_{\mathrm{s}} \\
&\overset{(2)}{=} \bigg(\Big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\Big) \circ \mathcal{S}(\sigma^{\odot}_{\mathscr{D},\mathscr{D}} \circ \Delta^d)\bigg) \circ \iota_{\mathrm{s}} \\
&\overset{(2)}{=} \Big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\Big) \circ \sigma^{\odot}_{\mathscr{D},\mathscr{D}} \circ \Delta^d \\
&\overset{(3)}{=} (\varphi_1 \odot \varphi_2) \circ \Delta^d \\
&\equiv \varphi_1 \circledast \varphi_2,
\end{aligned}
$$

where in (1) the statement of Thm. 3, in (2) universal property of $\mathcal{S}(\,\cdot\,)$ and in (3) assertion of Prop. 1 have been used $\square$

- What is good definition for exponential series on $((\mathscr{D}^d)^*, \circledast)$? answer is for all $b \in \mathscr{D}^d$

$$(\exp_\circledast \varphi)(b) := \sum_{n=0}^{\infty} (\mathrm{D}(\varphi)^{\star n} \circ \iota_{\mathrm{s}})(b),$$

where $\mathrm{D}(\varphi) \colon \mathrm{S}(\mathscr{D}^d) \to \mathbb{C}$ defined on $\mathrm{S}(\mathscr{D}^d) = \bigoplus_{n=0} \mathrm{S}^n(\mathscr{D}^d)$ with $\mathrm{S}^0(\mathscr{D}^d) = \mathbb{C}$

$$\mathrm{D}(\varphi)\restriction_{\mathrm{S}^n(\mathscr{D}^d)} := \begin{cases} 0 & \text{if } n = 0 \text{ or } n > 1 \\ \varphi & \text{if } n = 1. \end{cases}$$

- justified by the following:

Definition 10 (Convolution semigroup [BS05][3])

- if $(\mathscr{D}, \Delta, \delta)$ is comonoid in $(\mathsf{alg_m}, \sqcup)$, then family $(\varphi_t)_{t \in \mathbb{R}_+} \subseteq (\mathscr{D}^d)^*$ is called a convolution semigroup on $(\mathscr{D}, \Delta, \delta)$ if

$$\forall\, s, t \in \mathbb{R}_+ \colon \varphi_s \circledast \varphi_t = \varphi_{s+t} \text{ and } \varphi_0 = \delta^d \equiv \mathsf{0}$$

- convolution semigroup is weakly continuous if

$$\forall\, b \in \mathscr{D}^d \colon \lim_{t \to 0+} \varphi_t(b) = \delta^d(b) = 0.$$

[3] A. Ben Ghorbal and M. Schürmann. "Quantum Lévy processes on dual groups". In: *Math. Z.* 251.1 (2005), pp. 147–165.

Theorem 4 (Characterization of convolution semigroup on comonoid in $\mathsf{alg_m}$ [BS05, Thm. 4.6]) if $(\mathscr{D}, \Delta, \mathsf{0})$ is comonoid in $(\mathsf{alg_m}, \sqcup)$ and $(\varphi_t)_{t\in\mathbb{R}_+} \subseteq (\mathscr{D}^d)^*$ is convolution semigroup on $\mathscr{D}$. Following assertions are equivalent

i.) convolution semigroup $(\varphi_t)_{t\in\mathbb{R}_+}$ is weakly continuous.

ii.) $\exists \Psi \in (\mathscr{D}^d)^*$ such that $\forall t \in \mathbb{R}_+ \colon \varphi_t = \exp_\star(t\,\mathrm{D}(\Psi))\restriction_{\mathscr{D}}$

Ψ uniquely determined by $(\varphi_t)_{t\in\mathbb{R}_+}$, i.e.

$$\forall b \in \mathscr{D} \colon \Psi(b) = \lim_{t\to 0^+} \frac{\varphi_t(b)}{t}.$$

- we obtain for $\varphi \in (\mathscr{D}^d)^*$ and $b \in \mathscr{D}^d$

$$(\exp_{\circledast} \varphi)(b) := \sum_{n=0}^{\infty} (\mathrm{D}(\varphi)^{\star n} \circ \iota_{\mathrm{s}})(b)$$

seems good definition!

Central Limit Theorem for General Universal Products

- if $(V, (V^{\langle\alpha\rangle})_{\alpha\in\mathbb{N}_0})$ is $\mathbb{N}_0$-graded vector space and choose any $z \in \mathbb{C}$, then for homogeneous $v \in V$

$$S_z \colon V \ni v \mapsto z^{\deg v} v \in V$$

Theorem 5 (Central limit theorem for comonoids in $\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0}$ [Lac15, Thm. 7.1.2])

- $\odot$ is u.a.u.-product in $\mathsf{algP}_{\mathsf{d,m}}$
- comonoid $(\mathscr{D}, \Delta, \delta)$ in $\mathsf{alg}_\mathsf{m}^{\mathbb{N}_0})$ with induced $\mathbb{N}_0$-gradation denoted by $(\mathscr{D}^{\langle\alpha\rangle})_{\alpha\in\mathbb{N}_0}$
- $\varphi \in (\mathscr{D}^d)^*$ fullfills

$$\forall \alpha \text{ with } 0 \leq \alpha < \nu \colon \varphi\restriction_{(\mathscr{D}^d)^{\langle\alpha\rangle}} = 0,$$

$\Rightarrow$

$$\forall b \in \mathscr{D}^d \colon \lim_{n\to\infty} \left(\varphi^{\circledast n} \circ S_{n^{-\frac{1}{\nu}}}\right)(b) = (\exp_\circledast(g_\varphi))(b)$$

where $g_\varphi \in (\mathscr{D}^d)^*$ defined by

$$g_\varphi\restriction_{(\mathscr{D}^d)^{\langle\alpha\rangle}} = \begin{cases} \varphi\restriction_{(\mathscr{D}^d)^{\langle\nu\rangle}} & \text{if } \alpha = \nu \\ 0 & \text{otherwise} \end{cases}$$

Outlook

- try to calculate right hand side of central limit theorem, i.e. $(\exp_{\circledast}(g_\varphi))(b)$ for "interesting examples" of cases of u.a.u.-product, where intersting examples are given in motivation

Thank you very much for your attention!

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