The Steiner Ratio of Metric Spaces

A Report

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Contents

Chapter 1

Introduction

> Philosophy is written in this grand book of the universe, which stands continually open to our gaze.... It is written in the language of mathematics.
>
> Galileo Galilei

On March 19., 1836 the astronomer Schuhmacher wrote a letter to his friend, the mathematician Gauß, in which he expressed surprise about a specific case of Fermat's problem: He considered four points v_1, v_2, v_3, v_4 in the plane which forms a quadrilateral such that the segments $\underline{v_1v_2}$ and $\underline{v_3v_4}$ are parallel and the lines of the segments $\underline{v_1v_3}$ and $\underline{v_2v_4}$ meet in one point v outside. He considered the so-called Torricelli point q which is the point such that the sum of its distances to the given points is minimal. q is the intersection point of the diagonals $\underline{v_1v_4}$ and $\underline{v_2v_3}$. Schuhmacher did not understand the fact that, if the segment $\underline{v_3v_4}$ runs to the point v then the point q runs in the same way to v, but this cannot be, since the Torricelli point of three points is not necessarily one of the given points. Gauß [165] answered on March 21. that Schuhmacher did not consider Fermat's problem; instead, he looked for a solution of

$$||v_1 - w|| + ||v_2 - w|| + 2 \cdot ||v_3 - w|| = \text{min!} \tag{1.1}$$

More important was the next remark of Gauß. He said that it is natural to consider the following, more general question

> Ist bei einem 4Eck ... von dem kürzesten Verbindungssystem die Rede ..., bildet sich so eine recht interessante mathematische Aufgabe, die mir nicht fremd ist, vielmehr habe ich bei Gelegenheit eine Eisenbahnverbindung zwischen Harburg, Bremen, Hannover, Braunschweig...in Erwägung genommen ....

In English: "How can a railway network of minimal length which connects the four German cities Bremen, Harburg (today part of the city of Hamburg), Hannover, and

Braunschweig be created?"[1,2]
The problem seems disarmingly simple, but it is rich with possibilities and difficulties, even in the simplest case, the Euclidean plane. This is one of the reasons that an enormous volume of literature has been published, starting in the seventeenth century and continuing today.[3]
The history of the problem of "Shortest Connectivity" started with Fermat [147] early in the 17th century and Gauß [165] in 1836.[4] At first perhaps with the book *What is Mathematics* by Courant and Robbins in 1941 [110], this problem became popularized under the name of Steiner[5]:

> For a given finite set of points in the plane, find a network which connects all points of the set with minimal length.

Such a network must be a tree, which is called a Steiner Minimal Tree (SMT). It may contain vertices other than the points which are to be connected. Such points are called Steiner points. Given a set of points, it is *a priori* unclear how many Steiner points one has to add in order to construct an SMT.

Until 1961 it was not even known that Steiner's Problem is finitely solvable. There are infinitely many points in the plane, and even though most of them are probably irrelevant, it is not obvious that any algorithm exist. Then Melzak [266] gave a finite algorithm using a set of Euclidean (that is ruler and compass) constructions.[6]
A classical survey of Steiner's Problem in the Euclidean plane was presented by Gilbert and Pollak in 1968 [167] and christened "Steiner Minimal Tree" for the shortest interconnecting network and "Steiner points" for the additional vertices.

Over the years the problem has taken on an increasingly important role, not only in pure mathematics. Steiner's Problem arise in a wide range of application domains. More and more real-life problems are given which use it or one of its relatives as an application, as a subproblem or as a model. This tremendous interest in location modeling is the result of several factors. Today we can say that Steiner's Problem

[1] A picture of this letter can be found on the cover of the book *Approximation Algorithms* [360].

[2] In 1879 Bopp [40] discuss this four-point question, including a presentiment of its hardness for more points.

[3] If the reader will be new in this area of research: Find the shortest tree interconnecting the four vertices of a square of edge length 1. Hint: The network that joins all vertices to the center of the square is not a solution, [234].
Furthermore, a pleasant exposition is to look for shortest trees for points positioned on a rectangular grid, e.g. for the 81 points at the "corners" of a checkerboard, see [64], [159], [160] or [161].

[4] Martini in [38] names an older source, namely Lame and Clapeyron in 1827, but he doesn't give an exact reference. Scriba and Schreiber [323] give a discussion of the origins of the problem.

[5] A Swiss mathematician who lived from 1796 until 1863; although he apparently had nothing to do with Steiner's Problem, Kuhn, compare [372]: "Although this very gifted geometer (Steiner) of the 19th century can be counted among the dozens of mathematicians who have written on the subject, he does not seem to have contributed anything new, either to its formulation or its solution."
It seems that Courant and Robbins knew of a report by Steiner on "Fermat's Problem"(!) to the Prussian Academy of Sciences in 1837.

[6] Surprisingly, the Melzak algorithm cannot be extended to higher-dimensional Euclidean spaces, not even to spaces of dimension three.

and its relatives are one of the most famous combinatorial-geometrical problems next to the traveling salesman problem. This is not a surprise, Weber [369]:

> Wenn schon einmal Theorie getrieben werden soll, ... so ist als eine ihrer Formen auch diejenige nötig, die die Abstraktion auf die Spitze treibt.

In concise English: If you consider a problem, think about all consequences.[7]
In the introduction to the first issue of the journal *Location Science* the editors wrote:

> First, location decisions are frequently made at all levels of human organization from individuals and households to firms, governments, and international agancies. Second, such decisions are often strategic in nature; that is, they involve significant capital resources and their economic effects are long term in nature. Third, they frequently impose economic externalities. Such externalities include economic development, as well as pollution and congestion. Fourth, location models are often extremely difficult to solve, at least optimally. Even some of the most basic models are computationally intractable for all but the smallest problem instances. In fact, the computational complexity of location models is a major reason that the widespread interest in formulating and implementing such models did not occur until the advent of high speed digital computers. Finally, location models are application specific. Their structural form, "the objectives, constraints and variables", is determined by the particular location problem under study. Consequently, there does not exist a general location model that is appropriate for all, or even most, applications.

Smith [332] presents a classification of applications for network design problems. Generalizing this there are the following practical examples of Steiner's Problem.

- We saw that Gauß are interested in the question to link cities by railroads.
- Consider inter-urban networks, like communication networks, railway lines and interstate highway networks. The solution of network design problems in this area, whether approximate or exact, can provide guidelines for the layout of the network and the necessary amounts of material, [256], [257], [371].
- Design of computer chips. In VLSI placement, one optimizes the position of the modules of a chip such that the total interconnecting length becomes short, [230], [248], [272].
- Devoted to the development of robotics extremal networks becomes importance, [154].

[7] In this sense it is strange that people "discover" Steiner's Problem again and again, and prove "facts" which have already been proven a dozen times. One of these discoveries is the fact that the degree of a Steiner point in an SMT in Euclidean spaces of arbitrary dimension equals 3.

- Chemical processing plants, urban arterial systems, cable television and similar intra-urban systems are typical applications and electric, heating and air-conditioning systems in buildings. Often connection structures have to be designed in an environment with pronounced inner structure, [202], [244], [263], [330].
- Practical applications arise for example in the design of telecommunication networks which can "survive" certain edge failures, [139], [226].
- Location of international headquarters or distribution centers and planning of oil or natural gas pipelines or long distance telephone lines, [308], [312], [348].
- Underground mining industry, [44], [46], [370].
- Minimal Surfaces: There are many similarities between Steiner's Problem and minimal surfaces, which are helpful to attack these problems, [277]. In particular: soap films, [110], [111], [243], [286].[8]
- One of the key issues in biochemistry is predicting the three-dimensional structure of proteins from the primary sequence of amino acids, [273], [274], [275], [334], [353].
- To consider the problem of reconstruction the evolutionary history (phylogenetic trees), [100], [152], [153], [328], [351], [317]. Roughly spoken: A phylogenetic tree is a shortest one in a desired chosen metric space, [178], [88], [100].
- The anthropologists compare the species tree and the tree of languages for human populations, [56].
- The history of manuscripts dealt with reconstructing the copying procedure, [49], [270].
- A relational database can be described as a graph. Then the problem of finding a subtree interconnecting several items is Steiner's Problem in this graph, [16].

However, all investigations show the great complexity Steiner's Problem, as well in the sense of structural as in the sense of computational complexity. On the other hand, a Minimum Spanning Tree, this is a shortest tree interconnecting a finite set of points without Steiner points, can be found easily by simple and general applicable methods. In this sense, we define the Steiner ratio for a metric space (X, ρ) to be the greatest lower bound over all finite sets of points of the length of a Steiner Minimal Tree (SMT) divided by the length of a Minimum Spanning Tree (MST):

$$m(X, \rho) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \subseteq (X, \rho) \text{ is a finite set} \right\}.$$

[8] A specific question, the so-called "Double Soap Breakthrough", is ranked by the *Encyclopedia Britannica* as second only to Wiles' proof of Fermat's Last Theorem, [280].

This quantity is a parameter of the considered space and describes the performance ratio of the the approximation for Steiner's Problem by a Minimum Spanning Tree.[9] Steiner's Problem occupies a central place in the field of approximation algorithms.

This present book concentrates on investigating the Steiner ratio. The goal is to determine, or at least to estimate, the Steiner ratio for many different metric spaces. There are a very large number of metric spaces, such that a general and closed theory cannot be expected, hence we will concentrate on the spaces which are of practical interest.

The book started with some general assertions about Steiner's Problem, most of these are folklore. Then we investigate the most common and well-investigated spaces, namely the finite-dimensional normed spaces. Then it will be go further to infinite-dimensional ones and to metric spaces in general. Using these facts we discuss specific cases: Finite and discrete metric spaces and manifolds. At the end, we consider relatives of Steiner's Problem.

[9] We denote the Steiner ratio by the letter "m" in view of its role as a (geometric) measure.

Chapter 2

Steiner's Problem

Steiner's Problem is the problem of "Shortest Connectivity": Given a finite set of points in a metric space, search for a network that connects these points with the shortest possible length. Note that we look for the shortest network overall, that means, we have the freedom to introduce an undetermined number of new branching points everywhere in the space.

2.1 Trees

A graph G is defined to be a pair (V, E) where V is a nonempty and finite set of elements, called vertices, and E is a finite family of elements which are unordered pairs of vertices, called edges.
A key notion in graph theory is that of a connected graph. It is intuitively clear what this should mean: A graph $G = (V, E)$ is called a connected graph if for any two vertices there is a path interconnecting them. A tree is defined to be a connected graph without cycles. A vertex with degree one is called a leaf. A vertex in a tree that is not a leaf is called an internal vertex. Considering a longest path in a tree, then its endvertices must be leaves. Consequently,

Theorem 2.1.1 *Each tree with more than one vertex has at least two leaves.*

The following theorem establishes several of the most useful characterizations of a tree. Each contributes a deeper understanding of the structure of this basic type of graphs. In our further investigations we will use these equivalences permanently.

Theorem 2.1.2 *Let $G = (V, E)$ be a graph with n vertices, where $n > 1$.*[1] *Then the following properties are pairwise equivalent (and each characterizes a tree):*

- *G is connected and has no cycles.*
- *G is connected and contains exactly $n - 1$ edges.*

[1] By definition a graph with one vertex and without edges is also a tree.

- *G has exactly $n-1$ edges and has no cycles.*
- *G is maximally acyclic; that means G has no cycles, and if a new edge is added to G, exactly one cycle is created.*
- *G is minimally connected; that means G is connected, and if any edge is removed, the remaining graph is not connected.*
- *Each pair of vertices of G is connected by exactly one path.*

The *proof* uses induction and 2.1.1.

□

Let $T = (V, E)$ be a tree with n vertices. n_i denotes the number of vertices of degree i and $\Delta = \Delta(T)$ the maximum degree in T. Then, of course,

$$n_1 + n_2 + \ldots + n_\Delta = n. \tag{2.1}$$

In view of the double counting principle and 2.1.2, we have

$$n_1 + 2 \cdot n_2 + \ldots + \Delta \cdot n_\Delta = 2|E| = 2n - 2. \tag{2.2}$$

Subtracting this equation from two times (2.1) yields

Theorem 2.1.3 *It holds that*

$$n_1 = 2 + \sum_{i=3}^{\Delta(T)} (i-2) \cdot n_i, \tag{2.3}$$

for any tree T, where n_i denotes the number of vertices of degree i and $\Delta(T)$ is the maximum degree in the tree.

Consequently,

a) Considering only trees without vertices of degree two, the number of internal vertices is less than the number of leaves and a binary tree has the maximum possible number of internal vertices for a given number of leaves.

b) Each tree T with more than one vertex has at least $\Delta(T)$ leaves.

c) Trees can be generated recursively by appending repeatedly leaves starting with one vertex and vice versa leads to an elimination scheme where repeatedly leaves are deleted.

To count trees we have to distinguish between labeled and unlabeled ones. A tree $T = (V, E)$ is called labeled if there is a bijective mapping from V onto a set of $|V|$ distinct names in such a way as to be they are distinguishable from each other. With most enumeration problems, counting the number of unlabeled things is harder than counting the number of labeled things. So it is with trees.

Theorem 2.1.4 *Let $n \geq 2$ be an integer and let $g_1, ..., g_n$ be a sequence of positive integers. When we denote by $t(n, g_1, ..., g_n)$ the number of different labeled trees $T = (\{v_1, ..., v_n\}, E)$ of n vertices with the degree sequence $g_T(v_i) = g_i$ for $i = 1, ..., n$, we have*

$$t(n, g_1, ..., g_n) = \begin{cases} \frac{(n-2)!}{\prod_{i=1}^{n}(g_i-1)!} = \binom{n-2}{(g_1-1)...(g_n-1)} & : \sum_{i=1}^{n} g_i = 2n-2 \\ 0 & : \text{otherwise} \end{cases}$$

For a *proof* see [28].[2]

To count unlabeled graphs is the enumerating of isomorphic classes. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a one-to-one, onto mapping $f : V_1 \to V_2$ such that $\underline{vv'} \in E_1$ if and only if $\underline{f(v)f(v')} \in E_2$. f is called an isomorphism. In general it is difficult to determine whether two graphs are isomorphic.[3] Isomorphism is an equivalence relation on the collection of all graphs. A isomorphic class is also called the topology of the tree.

Theorem 2.1.5 *Let $T(n)$ be the number of non-isomorphic trees with n vertices. Then*

$$T(n) \geq \frac{n^{n-2}}{n!} \geq \frac{e^n}{en^{5/2}}. \tag{2.4}$$

On the other hand,

$$T(n) \leq \frac{1}{n}\binom{2n-2}{n-1} \approx \frac{4^n}{\sqrt{2\pi \cdot n^3}}. \tag{2.5}$$

For a *proof* compare [103].[4]

Further graph theoretic terminology and statements are given in most standard textbooks, for example [37], [188] or [174].

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[2]Summing up over all degree sequences gives one of the most beautiful formulas in enumerative combinatorics, namely Cayley's tree formula [57], that the number of different labeled trees with n vertices equals n^{n-2}. For several other proofs compare [2].

[3]It is strange, but its computational complexity is still unknown: No polynomially bounded algorithm is known, on the other hand it has not been proved that this problem is $\mathcal{NP}$-complete. A monograph on isomorphism detection is given in [193]. On the other hand, for trees the isomorphic problem is easy: there is a quadratic time algorithm which decides whether two trees are isomorphic; see [359]. But this does not mean that it is easy to count the number of such trees.

[4]All together we expect that $T(n) = \Theta(a^n/f(n))$ with $e \leq a \leq 4$ and a function $f(n)$ which is bounded by a low degree polynomial. And indeed, the number of unlabeled trees is asymptotically completely determined, since Pólya, compare [189] shows

$$T(n) \approx \frac{c \cdot a^n}{n^{5/2}}, \tag{2.6}$$

where $a = 2.9557\ldots$ and $c = 0.5349\ldots$.

2.2 Steiner's Problem in Metric Spaces

Solutions of Steiner's problem depend essentially on the way in which the distances in space are determined. In recent years it turned out that in network design many distances play an important role. This is not a surprise, since distance is the mathematical description of the idea of proximity, and this may take many different forms depending on the application.[5] In general, it is mentioned that Menger [268] already considered Steiner Minimal Trees in general metric spaces in 1931.

A metric space (X, ρ) is characterized by a set X of points equipped by a function $\rho : X \times X \to I\!R$ satisfying:

(i) $\rho(x, y) \geq 0$ for all x, y in X;

(ii) $\rho(x, y) = 0$ if and only if $x = y$;

(iii) $\rho(x, y) = \rho(y, x)$ for all x, y in X (symmetry); and

(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all x, y, z in X (triangle inequality).

Usually, such a function ρ is called a metric.[6,7] We will say that the quantity $\rho(x, y)$ is the distance between the points x and y.
A metric ρ on a finite set X of n points can be specified by an $n \times n$ matrix of (nonnegative) real numbers. Actually $\binom{n}{2}$ numbers suffice because of the symmetry of ρ and that the numbers on the main diagonal equals zero.

We consider **Steiner's Problem of Minimal Trees** by the following question:

Given: A finite set N of points in the metric space (X, ρ).

[5] Plastria [294]:"For example, in a mountainous region it happens frequently that one can easily communicate verbally between two places across a chasm, whereas moving physically from one place to another may call for a large detour because of lack of wings." Furthermore, the symmetry of the distance is not necessarily satisfied. Cook et al. [109] discussed the question "What would it be like to live in a space with a non-Euclidean norm, where length depends on direction?"

[6] The axioms are not independent: (i) is a consequence of the others. On the other hand, we can replace these collections of axioms equivalently by

(ii) $\rho(x, y) = 0$ if and only if $x = y$; and

(iv') $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ for all x, y, z in X.

[7] The following variants of "metric approaches" will be also of interest:

- If ρ satisfies (ii) only in the weaker form

  (ii') $\rho(x, x) = 0$ for all x in X;

  we say that ρ is a pseudometric.
- If the function ρ satisfies the conditions (i),(ii') and (iii) it is called a dissimilarity.

In both cases we can produce a metric. In the first, we introduce an equivalence relation by $x \sim y$ if and only if $\rho(x, y) = 0$; in the second we consider the metric closure.

Find: A connected graph $G = (V, E)$ embedded in the space such that $N \subseteq V$, and the quantity, called the length,

$$L(G) = L(X, \rho)(G) = \sum_{\underline{vv'} \in E} \rho(v, v') \tag{2.7}$$

is as minimal as possible.[8]

A solution must be a tree, called a Steiner Minimal Tree (SMT). An SMT may have some points not in N, which are called Steiner points. What makes the problem difficult is that we do not know *a priori* the location and the number of Steiner points.

In the last six decades the investigations and, naturally, the publications about Steiner's Problem have increased rapidly. A large literature has arisen trying to understand many different aspects of this problem. More and more applications were found and theoretical approaches were discussed. Surveys about Steiner's Problem in the form of monographs are necessary and were given by

1. S.Voß: "Steiner-Probleme in Graphen", 1990, [361].
2. F.K.Hwang, D.S.Richards, P.Winter: "The Steiner Tree Problem", 1992, [202].
3. A.O.Ivanov, A.A.Tuzhilin: "Minimal Networks - The Steiner Problem and Its Generalizations", 1994, [206].
4. D.Cieslik: "Steiner Minimal Trees", 1998, [83].
5. A.O.Ivanov, A.A.Tuzhilin: "Branching Solutions to One-Dimensional Variational Problems", 2001, [209].
6. D.Cieslik: "The Steiner Ratio", 2001, [92].
7. X.Cheng and D.Z.Du (eds.): "Steiner Tress in Industry", 2001, [61].
8. H.J.Prömel, A.Steger: "The Steiner Tree Problem", 2002, [301].
9. A.O.Ivanov, A.A.Tuzhilin: "Theory of Extreme Networks" (Russian), 2003, [210].
10. D.Cieslik: "Shortest Connectivity", 2005, [100].
11. D.Z.Du, X.Hu: "Steiner Tree Problems in Computer Communication Networks", 2008, [138].

There are several collections about Steiner's Problem and its relatives: [35], [136], [168], [190], [201], [208], [290], [360] and [363].[9] A representation of the complete subject, including its history, has been given in [31], [32], [69], [191], [205], [322] and [348].

[8] Sometimes it is useful to consider graphs as topological spaces glued from segments each of which corresponds to an edge of the graph. Then the edges are continuous curves in the ambient space.

[9] A very interesting observation: In each of the monographs and papers there is at least one aspect of Steiner's Problem which is not in the union of the others.

2.3 Minimum Spanning Trees

The minimum spanning tree problem is one of the most typical problems of combinatorial optimization; methods for its solution have generated important ideas of modern combinatorics and have played a central role in the design of computer algorithms. The problem is usually stated as follows:

> Given a weighted (connected) graph one would then wish to select for construction a set of communication links that would connect all the vertices and have minimal total cost.

In geometric terms: If we don't allow Steiner points in a shortest tree, that is if we connect certain pairs of given points only, then we refer to a Minimum Spanning Tree (MST).
Starting with Boruvka in 1926, Kruskal in 1956 and Prim in 1957, Minimum Spanning Trees have a well-documented history [170] and effective constructions [62], [234]. In view of the many contributions to the problem of constructing minimum spanning trees, its popularity through the ages, and its natural applications to various practical questions, it is hopeless to expect a complete list of the many facets of the problem. In other terms, the problem has an interest in its own.[10]

A minimum spanning tree in a graph can be found with the help of Kruskal's method[11]:

Algorithm 2.3.1 *(Kruskal [235]) A minimum spanning tree in a graph $G = (N, E)$ with a positive length-function $f : E \to I\!R$ can be found*

1. *Start with the forest $T = (N, \emptyset)$;*
2. *Sequentially choose the shortest edge that does not form a circle with already chosen edges;*
3. *Stop when all vertices are connected, that is when $|N|-1$ edges have been chosen.*

Then an MST for a finite set N of points in a metric space (X, ρ) can be found obtaining the graph $G = (N, \binom{N}{2})$ with the length-function $f : \binom{N}{2} \to I\!R$ given by $f(\underline{vv'}) = \rho(v, v')$.
Kruskal's algorithm finds an MST for n points in $O(n^2 \log n)$-time.[12]

[10] It seems to be the first network optimization problem ever studied. Its history dates back to at least 1926. Boruvka [42] produced the first fully realized minimum spanning tree algorithm by a parallel technique, and it has been rediscovered several times, Sollin in [27]

[11] This cheapest-link algorithm is the mother of all greedy algorithms, that is to takes the best choice and run, [231], [234].
Another method, created by Prim [300] and Dijkstra [114], is a typical example for dynamic programming.

[12] There are several minimum spanning tree algorithms for graphs that are asymptotically faster: Yao [381], Gabow et al. [157].

Remark 2.3.2 *([223]) The time complexity to find an MST in a metric space is of order* $\Theta(n^2)$.[13]

A complete discussion of minimum spanning tree strategies in networks are given by [349], [350], [377].

2.4 Fermat's Problem

We discuss the following geometric optimization problem, since it is the local version of Steiner's Problem, but, of course, it has an interest on itself.
Let N be a finite set of points in a metric space (X, ρ). Determine a point w in the space such that the function

$$F_N(w) = \sum_{v \in N} \rho(v, w) \tag{2.8}$$

is minimal. It is called Fermat's Problem.[14] Note that in the case when the number of given points equals three, then Steiner's and Fermat's Problem coincide with each other.
Each point which minimizes the function F_N is called a Torricelli point for N in (X, ρ). For specific spaces (X, ρ) and special sets N a Torricelli point can be not unique. Then $L_N(X, \rho)$ denotes the set of all. It is a bounded set:

$$L_N(X, \rho) \subseteq \{x \in X : \rho(v, x) \leq (|N| - 1) \cdot D(N)\}, \tag{2.9}$$

where $v \in N$ is a given point and $D(N) = \max\{\rho(v, v') : v, v' \in N\}$ denotes the diameter of N.[15]

The problem has a long and strange history; moreover, it has gone by many names. Although the ancient Greeks knew that the shortest path connecting two points was a straight line, it was apparently Fermat who first asked what the shortest path was connecting three points. He posed the problem early in the 17th century at the end of his book *Treatise on Minima and Maxima* [147] and was stated exactly as follows:

> Given three points in the plane, find a fourth point such that the sum of its distances to the three given points is minimal.

[13] The problem of finding an MST for a set of points in an affine space differs from the problem of finding a minimum spanning tree in a general metric space in the following sense: The input consists of the numbers which describe the coordinates of the points, the edges and their lengths being implicitly defined by an analytical system. Hence, it is useful and interesting to consider if the geometric nature of the problem can be exploited to obtain fast algorithms for finding an MST. So, it is not astonishing that the time to find an MST in such a space is substantially shorter than the time $\Theta(n^2)$. For example $O(n \log n)$ in planes with p-norms, Lee [245].

[14] And often, but incorrectly, "Steiner-Weber" Problem, coming from the popular book [369] and its appendix.

[15] As an extreme case for the set $N = \{(\pm 1, 0), (0, \pm 1)\}$ in the rectilinear plane L_N coincides with the convex hull of N.

Players from a lot of fields of study have stepped on its stage, and some of them have stumbled. It is usual to credit the Italian mathematicians with proposing and solving the problem. Around 1640 Torricelli solved this problem: He asserted that, assuming that the given points form a triangle in which all angles are less than 120^o, the circles which circumscribe the equilateral triangles constructed on the sides of and outside of the given triangle intersect in the desired point.
In the following centuries this problem was well established in the mathematical folklore. Probably, the first generalizations were given by Simpson in 1750 and then by Steiner in 1837 and Weber in 1909. It is natural to generalize Fermat's problem to a finite set of points in metric spaces.[16]
Solutions to Fermat's problem depend fundamentally on the way in which distances in the space are determined. Consequently, there are many metric spaces to be considered. A general strategy to determine the Torricelli points is unknown, and it is not to expect that such methods will be created, since the class of all metric spaces is too wide. In other terms, solution methods and the complexity of solving Fermat's problem will be fundamentally different; there is no common technique.[17] Due to the

[16]A history of Fermat's Problem can be found in [38], [206], [236], [323], and [372].

[17]**I.** In the Euclidean plane Fermat's Problem is not simple:

Example 2.4.1 *For a small number of points there are geometric constructions, that is by usage of ruler and compass, to find a Torricelli point:*

a) *(Torricelli 1646, Cavalieri 1647, compare [283]) Let $n = 3$. If the convex hull of N forms a triangle in which each angle is less than 120^o, then the Torricelli point for $N = \{v_1, v_2, v_3\}$ can be found with the following construction: Find an equilateral triangle for v_1, v_2, v' drawn along one side with the third node v'; Construct the circle C circumscribing the triangle; The Torricelli point is the point where the segment $\overline{v_3v'}$ intersects the circle C.*
Otherwise, if one of the angles is at least 120^o, one of the given points is the Torricelli point, namely the point in which this angle is present.

b) *(Fagnano [144]) Let $n = 4$. If N forms a convex quadrilateral then the Torricelli point is the intersection of the diagonals of that quadrilateral. Otherwise, one of the points of N is the Torricelli point, namely the point within the convex hull of N.*

c) *(Bajaj [17], Mehlhos [265]) But, for $n \geq 5$ such a method does not exist.*

Mehlhos [265] further shows that in general it is impossible to construct a Torricelli point for 4 or more points in the three-dimensional Euclidean space by ruler and compass. (For a general introduction into Galois' theory see [13] or [337].)

II. In the sequence space (A^d, ρ_H) with alphabet A and Hamming distance ρ_H Fermat's Problem is extremely simple to solve: A Torricelli point for a set of given words can be found by the so-called majority rule, which says that for each coordinate we choose the letter of A which appears most frequently in this coordinate of the given words. As an example consider the following English words of length 5:

w_1	M	E	L	O	N
w_2	M	A	N	G	O
w_3	H	O	N	E	Y
w_4	S	W	E	E	T
w_5	C	O	O	K	Y
consensus	M	O	N	E	Y

practical importance of Fermat's Problem, the publications about this subject have increased rapidly. Surveys in the form of monographs are given by

1. W.Domschke, A.Drexl: "Logistik: Standorte", 1982, [116].
2. R.F.Love, J.G.Morris, G.O.Wesolowsky: "Facilities Location", 1989, [257].
3. H.W.Hamacher: "Mathematische Lösungsverfahren für planare Standortprobleme", 1995, [186].
4. D.Cieslik: "Steiner Minimal Trees", 1998, [83].
5. V.Boltjanski, H.Martini, V.Soltan: "Geometric Methods and Optimization Problems", 1999, [38].
6. A.Schöbel: "Locating Lines and Hyperplanes", 1999, [321].
7. Z.Drezner and H.W.Hamacher, (editors): "Facility Location", 2002, [119].

Collections about Fermat's Problem and its relatives in several metric spaces are given in [23], [59], [58], [70], [143], [191], [233], [239], [251], [261], [267], [281], [287], [293], [318], and [378].

2.5 Properties of SMT's

At first glance it seems that our two problems in the previous sections have not many facets in common. Fermat's problem is a typical one in the class of geometric, and the problem of a minimum spanning tree in the class of combinatorial optimization problems. And moreover, we used very different methods to find solutions. But together they are simplifications of the problem of "Shortest Connectivity". In this sense we start with a general analysis of Steiner's Problem in arbitrary metric spaces, describing several basic facts about the combinatorial and geometrical structure of SMT's, which are necessary to combine for finding a solution. Most of these facts are folklore; later we will discuss more detailed results that arise if we restrict ourselves to specific cases.

Observation 2.5.1 *A solution of Steiner's Problem cannot contains a cycle.*

Removing one edge from from a cycle of a connected graph does not destroy the connectivity. Therefore, repeating this procedure over and over again , we obtain an acyclic connected graph. That is a tree, called a Steiner Minimal Tree (SMT). We have the following properties of an SMT $T = (V, E)$ for a finite set N of given points in a metric space (X, ρ):

I. First, we have no isolated vetices, that is

Observation 2.5.2 *The degree of each vertex is at least one.*

All leaves of an SMT must be given points. Vertices in $V \setminus N$ are called Steiner points.

Observation 2.5.3 *Without loss of generality, we may assume that the degree of each Steiner point is at least three.*

Proof. It is impossible for a Steiner point v to have degree one, since the edge which joins v with the remaining tree has a positive length, and, therefore contradicts the minimality requirement.
A Steiner point of degree two can be eliminated using the triangle inequality without making the tree longer.

$\square$

Observation 2.5.4 *It is sufficient to consider only finite trees as candidates for an SMT.*

The *proof* uses only the first both observations above and elementary properties of trees, see [100].

$\square$

A Steiner point of degree two can be eliminated without lengthening the tree. For proof-technical reason a Steiner point v adjacent to w and w' are allowed, but then with $\rho(w, v) + \rho(v, w') = \rho(v, v')$.

Observation 2.5.5 *If $V \setminus N \neq \emptyset$, then there is a Steiner point which is adjacent to two given points.*

Proof. Let $T = (V, E)$ be an SMT for N. Assuming that each vertex in $Q = V \setminus N$ is adjacent to at most one vertex in N. The set Q induces in T a subgraph $G' = (Q, E')$, for which it follows

$$|E'| = \frac{1}{2} \sum_{v \in Q} g_{G'}(v) \geq \frac{1}{2} \sum_{v \in Q} (g_T(v) - 1) \geq \frac{1}{2} \sum_{v \in Q} 2 = |Q|.$$

This contradicts the fact that the forest G' has at most $|Q| - 1$ edges.

$\square$

Observation 2.5.6 *There are at most $|N| - 2$ Steiner points. Hence, the tree has at most $2|N| - 2$ vertices and $2|N| - 3$ edges.*

Proof.

$$2 \cdot |N| + 2 \cdot |V \setminus N| - 2 = 2 \cdot (|V| - 1) = 2 \cdot |E| = \sum_{v \in V} g_T(v) \geq 3 \cdot |V \setminus N| + |N|.$$

$\square$

II. When a given point v is not a leaf, the tree T can be decomposed (by splitting at v) into several smaller trees, so that given points only occur as leaves.

Observation 2.5.7 *Equality in 2.5.6 holds if and only if the tree is a binary one with exactly $|N|$ leaves, called a full tree.*

The *proof* follows immediately by the proof of 2.5.6.

$\square$

Observation 2.5.8 *A binary SMT $T = (V, E)$ for N consists of full subtrees which intersect only in vertices that are given points and of degree two or greater. The number of full subtrees of T is*

$$1 + \sum_{v \in N} (g_T(v) - 1). \tag{2.10}$$

III. We have exact values for the number of full trees:

Observation 2.5.9 *For the number of trees the following holds true:*

a) *The number of full trees with n labeled leaves and $n-2$ labeled internal vertices equals*

$$\frac{(2n-4)!}{2^{n-2}}. \tag{2.11}$$

b) *(Cavalli-Sforza, Edwards [54]) The number of full trees with n labeled leaves and $n-2$ unlabeled internal vertices equals*

$$(2n-5)!! = 1 \cdot 3 \cdot 5 \cdots (2n-5) = \Omega\left(\left(\frac{2n}{3}\right)^{n-2}\right). \tag{2.12}$$

Proof. a) We uses 2.1.4. Any tree has exactly n leaves and exactly $n-2$ vertices of degree 3. Then we count the number of trees by

$$\binom{(2n-2)-2}{\underbrace{(1-1)\ldots(1-1)}_{n-\text{times}}\underbrace{(3-1)\ldots(3-1)}_{(n-2)-\text{times}}} = \frac{(2n-4)!}{2^{n-2}}.$$

b) If the $n-2$ internal vertices are unlabeled, then this number must divided by $(n-2)!$. Thus

$$\begin{aligned} \frac{(2n-4)!}{2^{n-2}(n-2)!} &= \frac{(2n-4)\cdot(2n-5)\cdot(2n-6)\cdot(2n-7)\cdots 4\cdot 3\cdot 2\cdot 1}{2(n-2)\cdot 2(n-3)\cdots 2\cdot 2\cdot 2\cdot 1} \\ &= (2n-5)(2n-7)(2n-9)\cdots 5\cdot 3\cdot 1. \end{aligned}$$

$\square$

In other terms, Hall [185]:

Number of leaves	Number of binary trees	Comment
3	1	
4	3	
5	15	
10	10395	
22	$\approx 3 \cdot 10^{23}$	Almost a mole of trees
50	$\approx 3 \cdot 10^{74}$	More trees than the number of atoms in the universe
100	$\approx 2 \cdot 10^{182}$	out of any range

2.5.9 shows that the number of trees grows very rapidly in the number of vertices. For this fact and related questions compare [146] and [103].[18]

IV. Let $T = (V, E)$ be a tree. Assume that the vertices are labeled, i.e. $V = \{v_1, ..., v_n\}$. We define the adjacency matrix $A(T) = (a_{ij})_{i,j=1,...,n}$ with

$$a_{ij} = \begin{cases} 1 & : \text{ if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & : \text{ otherwise} \end{cases}$$

$A(T)$ contains the complete information about the structure of the tree T.[19]

Observation 2.5.10 *For a given topology of a tree T, given by its adjacency matrix $A(T) = (a_{ij})$, its length in a metric space is a linear function of the metric:*

$$L(X, \rho)(T) = \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} a_{ij} \cdot \rho(v_i, v_j). \tag{2.13}$$

V. Let N' be a subset of N, then an SMT for N connects also the points of N'. Hence,

Observation 2.5.11 *Let $N' \subseteq N$. Then*

$$L(SMT \text{ } for \text{ } N') \leq L(SMT \text{ } for \text{ } N). \tag{2.14}$$

Note, that a similar monotonicity property does not hold for MST's in general. Now, we will discuss the relation between the length of an SMT and an MST for the same finite set of points. By definition, an SMT cannot be longer than an MST: $L(\text{SMT for } N) \leq L(\text{MST for } N)$. On the other hand,

[18]This was one of the pessimistic view by Graham and Foulds [169], that it will be unlikely that minimal phylogenies for realistic number of "living entities" can be constructed in reasonable computational time. Today we are a little bit more optimistic. In particular by applying PAUP, which stands for "Phylogenetic analysis using parsimony"; see Hall [185] and Swofford [346].

[19]The adjacency matrix of a graph does depend on the labeling of the vertices; that is, a different labeling of the vertices may result in a different matrix, but they are closely related in that one can be obtained from the other simply by interchanging rows and columns.

Observation 2.5.12 *An SMT is an MST for the set $N \cup Q$, where Q is the set of Steiner points:*

$$L(SMT\ for\ N) = \inf\{L(MST\ for\ N \cup Q) : |Q| \leq |N| - 2\}. \tag{2.15}$$

Proof. If the Steiner points have been localized, an SMT for N is simple to find as the MST for all points. In view of 2.5.6 we may restrict the cardinality of Q.

□

VI. Note that it is possible that an SMT for a finite set does not exist. That means, there are metric spaces in which not every finite set has an SMT. (The incompleteness of the space can be the reason for this.) A simple example: Consider three points v_1, v_2 and v_3 which form the nodes of an equilateral triangle in the Euclidean plane. An SMT uses one Steiner point q, which is uniquely determined by the condition that the three angles at this point are equal, and consequently equal 120^o. Now, remove q from the plane, and we cannot find an SMT for v_1, v_2 and v_3 in this new metric space.

Observation 2.5.13 *(Cockayne [106]) An SMT for a finite set of points in a metric space (X, ρ) exits if the metric satisfies the following properties:*

1. *(X, ρ) is finitely compact.*
2. *There exists a geodesic in (X, ρ) joining each two points of X.*
3. *For $x, y \in X$, the quantity $\rho(x, y)$ is the length of a geodesic joining x and y.*

In particular, the conditions of the observation are satisfied by finite-dimensional Banach spaces and manifolds.[20] Nevertheless, in any case, the greatest lower bound $\inf\{L(\text{MST for } N \cup Q)\}$ does always exist. In what follows, this quantity is set as the length of a shortest tree, irrespective of the existence of an SMT for N.

VII. Where are Steiner points located? Steiner point locations in the space are not prespecified from a candidate list of point locations, but we may assume that the set of Steiner points is contained in a suitably bounded set:

Observation 2.5.14 *The set of all Steiner points is contained in $\{w : \rho(v, w) \leq L(MST\ for\ N)\}$, where v is a point of N.*

Secondly, Steiner points are the local solutions of Fermat's Problem:

Observation 2.5.15 *A Steiner point is a Torricelli point of its neighbors.*

Comparing all these facts, the search for an SMT for a finite set of points in a metric space forces investigations of two specific questions:

[20] For a general geometry of geodesics see Busemann [51].

- How many Steiner points are used in an SMT?[21]
- Where are these Steiner points located in the space?

Unfortunately, these questions cannot be solved independently from the construction of the shortest tree itself.[22]

VIII. Methods to find an SMT for N are still unknown or at least hard in the sense of computational complexity. For example, the algorithmic problem of finding an SMT of a given set of points is already $\mathcal{NP}$-hard in the Euclidean plane, but there are polynomial approximation schemes, see Arora [12], as well as exact algorithms that are feasible at least for up to several thousand points, see Warme et al. [367]. In particular for specific spaces:

space	complexity	source
Euclidean plane	$\mathcal{NP}$-hard	[162]
Rectilinear plane $\mathcal{L}_1^2$	$\mathcal{NP}$-hard	[163]
$\mathcal{L}_p$-planes	algorithm needs exponential time	[82]
Banach plane	algorithm needs exponential time	[79]
Graph	$\mathcal{NP}$-hard	[221]
Hypercube	$\mathcal{NP}$-hard	[151]
Phylogenetic space	algorithm needs exponential time	[96]
Grid	$\mathcal{NP}$-hard	[163]

For a complete discussion of these difficulties see [83] and [202].[23] This reinforces the interest and the recent emphasis on the development of polynomial-time approximations and heuristics for Steiner's Problem.

2.6 Ockham's Razor

The essentially scientific part in any theory is the mathematical one. The essence of the application of mathematics to any branch of science is the recognition and

[21] But note that the number of Steiner points is not unique determined. For example consider SMT's for the square $\{(\pm 1, \pm 1)\}$ in the rectilinear plane.

[22] For example, not every locally minimal tree is an SMT. Large-scale rearrangements of the Steiner points may be necessary to transform a network into a shortest possible tree, which is a globally minimal tree: Consider the four corners of a rectangle in the Euclidean plane measuring three units by four units. An MST for these points has length 10. There are two locally minimal trees with two Steiner points. Each arrangement forms a tree that has three edges connected to each Steiner point at 120^o. If the Steiner points are arranged parallel to the width, the locally minimal tree that results is $9.928\ldots$ units long. If the Steiner points are arranged parallel to the length, a locally minimal tree results with a length of $9.196\ldots$. Consequently, only in the last case do we have an SMT. Compare [132].

[23] For an introduction into complexity theory see [164] and for the spectrum of computational complexity see [349], [350].

exploitation of regularity, which may be rigid and striking or a dimly observed tendency hardly distinguishable amidst a general confusion.[24] Scientific or engineering applications usually require solving mathematical problems. This is indeed true for creating networks.
A general network design problem is for a given configuration of points to find a network which contains these objects, fulfilling some predetermined requirements and minimizes a given objective function. This is quite general and models a wide variety of network design problems of significant importance and nontrivial complexity. The network topology and design characteristics of these systems are classical examples of optimization problems, [101].

Science is a method, not simply a particular body of knowledge. In formulating hypotheses, scientists attempt to derive the simplest possible explanation that accounts for the data. Exactly this **Ockham's razor** says:

> **The best hypothesis is the one requiring the smallest number of assumptions.**

In other words, (more roughly spoken:) Keep it simple; (more exactly in Latin:) Entia non sunt multiplicanda praeter necessitatem.
With the "razor", Ockham cuts out all superfluous, redundant explanations. Note, that we do not use this principle for Steiner's problem in a simple sense[25]; that means that among all possible network structures we search one which satisfy only two restrictions:

- The network has to connect the given points. The concrete kind of the network is not predetermined.
- Only the total length of the network is minimized. This is obviously a natural demand in a metric space.

For a broader philosophical discussion of Ockham's razor see Brown [48] and Russell [313]. Hildebrandt and Tromba [191] give a nice introduction to this view of *The Parsimonious Universe*.[26]

[24] Roughly spoken: Mathematics is not a scientific theory; but without mathematics science is impossible.

[25] Described by Cavalli-Sforza [55] for the problem of phylogeny:

> ... it does not necessarily follow that a method of tree reconstruction minimizing the number of mutations is the best or uses all the information contained in the sequences. The minimization of the number of mutation is intuitively attractive because we know that mutations are rare. There may be some confusion, however, between the advantage of minimizing the number of mutations and sometimes invoked parallel of Ockham's razor ..., which was developed in the context of mediaval theology. The extrapolation of Ockham's razor to the number of mutations in an evolutionary tree is hardly convincing.

Semple and Steel [325] relate maximum parsimony and Steiner Minimal Trees in this case.

[26] On one hand, Steiner's Problem is a generalization of Hilbert's fourth problem of geodesics, [9], [380]. On the other hand, it is a specification of Plateau's problem of minimal surfaces, [111], [276], [278].

Chapter 3

The Steiner Ratio

Sometimes the Minimum Spanning Tree (MST) and the Steiner Minimal Tree (SMT), as the shortest network, are one and the same, but most of the time they are not. When they are not the same, the SMT is obviously shorter than an MST, but by how much?

3.1 The Interest in the Ratio

All investigations of Steiner's Problem show the great complexity of the problem, as well in the sense of structural as in the sense of computational complexity. In other terms:

Observation I.
In general, methods to find an SMT are hard in the sense of computational complexity or still unknown.[1] In any case we need a subtle description of the geometry of the space.

On the other hand,

Observation II.
It is easy to find an MST by an algorithm which is simple to realize and running fast in all metric spaces. The algorithm does not need any geometry of the space, it only uses the mutual distances between the points.

Shortly spoken: Steiner's Problem is in almost all cases a hard and difficult problem; but to find an MST is always simple. Thus we can use an MST to approximate an SMT, and therefore it is of interest to know what the performance ratio is. In this sense, we define the Steiner ratio as a parameter of a metric space (X, ρ) to be the

[1] Roughly spoken: All known exact algorithms to solve Steiner's Problem are in some way enumerative algorithms. However, they differ in how the enumeration is done and how clever their strategies are in avoiding total enumeration. Consequently, in view of 2.5.9, they need exponential time. Only in several specific metric spaces Steiner's Problem is simple.

largest lower bound over all finite sets of points of the length of an SMT divided by the length of an MST:

$$m(X, \rho) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \text{ a finite set in the space } X \right\}.$$

Roughly speaking, $m(X, \rho)$ says how much the total length of an MST can be decreased by allowing Steiner points:

$$L(X)(\text{SMT for } N) \geq m(X, \rho) \cdot L(X)(\text{MST for } N). \tag{3.1}$$

The quantity $m(X) \cdot L(X)(\text{MST for } N)$ would be a convenient lower bound for the length of an SMT for any set N in the metric space (X, ρ).[2]
In general, Steiner's Problem may fail to exist for some finite set in some specific metric spaces. Consequently, we define the Steiner ratio more carefully, namely, for set of points where an SMT exists:

$$m(X) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \text{ a finite set in } X \text{ and } L(\text{SMT for } N) \text{ defined by (2.15)} \right\}.$$

Hence, the interest in the Steiner ratio comes from two sources:

1. It is the approximation- (performance-) ratio of Steiner's Problem.
2. It is a measure of the geometry of a metric space related to its combinatorial properties.

3.2 The Steiner Ratio of Metric Spaces

It is obvious that $0 < m(X, \rho) \leq 1$ for the Steiner ratio of each metric space (X, ρ). For a metric space $m = 1$ the problem of finding an SMT trivially reduces to computing an MST. As an example, for the real line the MST and the SMT are identical, and its Steiner ratio equals 1. On the other hand, the lower bound can be given sharper:

Theorem 3.2.1 *(Moore in [167]) For the Steiner ratio of a metric space (X, ρ) it holds*

$$m(X, \rho) \geq \frac{1}{2}.$$

[2] We define the Steiner ratio as a relative approximation. An absolute one is senseless, since:

Observation 3.1.1 *(Widmayer [374]) Unless $\mathcal{P} = \mathcal{NP}$, no polynomial time approximation algorithm $\mathcal{M}$ for Steiner's Problem in networks can guarantee*

$$L(\mathcal{M}(N)) - L(SMT \text{ for } N) \leq K, \tag{3.2}$$

where N is a given set of vertices in the network, and K is some fixed constant.

Proof. Let T be an SMT for a finite set N. Consider the graph G obtained by replacing each edge of T by two parallel edges. Since an even number of edges is incident with each vertex of G, the graph G has a Eulerian cycle which has the length $2 \cdot L(T)$ and is a tour through N. This tour is not shorter than a minimal tour in which no Steiner point exists. If we delete any edge of the minimal tour, we obtain a tree interconnecting N without Steiner points. Hence,

$$L(\text{MST for } N) \leq 2 \cdot L(T) = 2 \cdot L(\text{SMT for } N), \tag{3.3}$$

implies the assertion.

□

The proof of 3.2.1 can be used to show a slightly stronger result, namely

Corollary 3.2.2 *Let N be a finite set of n points in a metric space (X, ρ). Then*

$$L(MST \text{ for } N) \leq 2 \cdot \left(1 - \frac{1}{n}\right) \cdot L(SMT \text{ for } N). \tag{3.4}$$

In 2.5.8 we decomposed SMT's. For the Steiner ratio this has the following consequence.

Theorem 3.2.3 *Let (X, ρ) be a metric space in which any Steiner point in an SMT has degree exactly three.*[3] *Then*

$$m(X, \rho) \geq m$$

holds if and only if

$$\frac{L(SMT \text{ for } N)}{L(MST \text{ for } N)} \geq m$$

for all full SMT's for N.

Proof. If an SMT is not a full tree, it may be decomposed into full subtrees T_i. Let T_i be a tree interconnecting the given points from N_i, and let T_i' be the MST for N_i. Then $\cup T_i'$ is a tree for N, perhaps not minimal, and having the length

$$\sum L(T_i') \geq L(\text{MST for } N). \tag{3.5}$$

On the other hand, it holds

$$\sum L(T_i) = L(\text{SMT for } N). \tag{3.6}$$

Both, (3.5) and (3.6), imply the asertion.

□

[3]For instance Euclidean spaces are of such kind.

For $n \geq 2$ we define

$$m^n(X, \rho) := \inf \left\{ \frac{L(SMT \text{ for } N)}{L(MST \text{ for } N)} : N \subseteq X, |N| \leq n \right\}, \tag{3.7}$$

Then, obviously, this quantity is monotonically decreasing in the value n: Starting with $m^2(X, \rho) = 1$ we have $m^{n+1}(X, \rho) \leq m^n(X, \rho)$ for $n > 2$; and consequently

$$\begin{aligned} m(X, \rho) &= \inf\{m^n(X, \rho) : n \text{ a positive integer}\} && (3.8) \\ &= \lim_{n \to \infty} m^n(X, \rho). && (3.9) \end{aligned}$$

Theorem 3.2.4 *For any metric space (X, ρ) it holds that*

$$m^3(X, \rho) \geq \frac{3}{4}.$$

Proof. Let an SMT for $N = \{v_1, v_2, v_3\}$ be given, which creates a star consisting of three edges from v_1, v_2 and v_3 to the common Steiner point v.
Say that $\rho(v_2, v_3)$ is greater than both $\rho(v_1, v_2)$ and $\rho(v_1, v_3)$. Then

$$L_M := L(\text{MST for } N) = \rho(v_1, v_2) + \rho(v_1, v_3).$$

The SMT for N has a length L_S less than L_M. Then

$$\begin{aligned} 4 \cdot L_S &= 4 \cdot (\rho(v_1, v) + \rho(v_2, v) + \rho(v_3, v)) \\ &= 2 \cdot (\rho(v_1, v) + \rho(v, v_2)) + 2 \cdot (\rho(v_2, v) + \rho(v, v_3)) \\ &\quad +2 \cdot (\rho(v_3, v) + \rho(v, v_1)) \\ &\geq 2 \cdot (\rho(v_1, v_2) + \rho(v_2, v_3) + \rho(v_3, v_1)) \\ &= 2L_M + 2\rho(v_2, v_3) \\ &\geq 2L_M + \rho(v_1, v_2) + \rho(v_1, v_3) \\ &= 3L_M. \end{aligned}$$

□

How we can extend this fact?[4]

We obtain two consequences:

[4]In normed planes X, Du et al. [131] show

$$m^n(X) \geq \frac{n}{2(n-1)}.$$

Proof. "Inflate" the edges of an SMT T for N to have the width ϵ. Thus, T becomes a polygonal region with a boundary. Suppose that $v_1, \ldots, v_n$ are the given points labeled in counterclockwise order on the boundary. Consider n spanning trees each of which is obtained by deleting an edge from the cycle $v_1, v_2, \ldots, v_n, v_1$. The total length of these n spanning trees is

$$(n-1) \cdot \text{ length of the cycle.} \tag{3.10}$$

Moreover, the length of the cycle is less than $2 \cdot L(T)$. Therefore, for an MST T' for N, we have

$$2(n-1) \cdot L(T) \geq n \cdot L(T'). \tag{3.11}$$

- To show that a metric space has Steiner ratio 3/4 or less, we need a four-point set.
- To show that a metric space has Steiner ratio 1/2, we need sets of arbitrary large cardinality.

3.3 The Achievement of the Steiner Ratio

It is often simple to determine an upper bound for the Steiner ratio of a specific space, since we only have to find a finite set of points with an interconnecting tree shorter than the MST.
We said that a (finite) set N_0 of points in a metric space (X, ρ) achieves the Steiner ratio if

$$\frac{L(\text{SMT for } N_0)}{L(\text{MST for } N_0)} = m(X, \rho) \tag{3.12}$$

Maybe such sets do not exist. But, if N_0 really exists, we have

$$m(X, \rho) = m^{|N_0|}(X, \rho).$$

Here, we define for a finite set N of points in (X, ρ) the quantity

$$\mu(N) = \mu(N)(X, \rho) = \frac{L(\text{SMT for } N)}{L(\text{MST for } N)}. \tag{3.13}$$

Obviously, $\mu(N) \geq m(X, \rho)$ and $m(X, \rho) = \inf\{\mu(N) : N \subseteq X\}$. For instance, consider the nodes of a regular polygon in the Euclidean plane: The perimeter of a regular n-gon without any side is an SMT for $n \geq 6$. In other terms,

Example 3.3.1 *For the nodes of a regular polygon (in the Euclidean plane) with at least $|N| \geq 6$ nodes it is $\mu(N) = 1$.*

Jarnik and Kösler [214] proved this result in 1934 for the regular hexagon, and for all regular n-gons with $n \geq 13$. It was another fifty years until Du, Hwang and Weng [124] showed it for all $n \geq 6$. For instance in [104] the cases $n = 3, 4, 5$ are discussed:

n	$\mu =$
$= 3$	$\frac{\sqrt{3}}{2} = 0.86602\ldots$
$= 4$	$\frac{\sqrt{3}+1}{3} = 0.91068\ldots$
$= 5$	$0.9945\ldots$
≥ 6	1

Du, Hwang, Chao [122] have shown that the SMT for a set of points on a circle is the shortest path connecting them if at most one distance between consecutive points is large enough.
The problem of achievement is not simple, also in the Euclidean case:

Conjecture 3.3.2 *a) The Steiner ratio of the Euclidean plane is achieved by a finite set of points, namely the nodes of an equilateral triangle.*
b) On the other hand, there does not exist a finite set of points in the 3-dimensional Euclidean space, which achieves the Steiner ratio.

An immediately consequence of 3.2.2 is

Corollary 3.3.3 *Let (X, ρ) be a metric space with Steiner ratio $1/2$.*[5] *Then there does not exist a finite set of points in X which achieves the Steiner ratio.*

3.4 Approximations and Heuristics

We have seen that for most spaces, all known deterministic methods for finding SMTs need exponential time. This reinforces the interest and the recent emphasis on the development of polynomial-time approximations and heuristics for Steiner's Problem.

Let N be a finite set of points in a metric space (X, ρ). Consider an approximation algorithm or a heuristic $\mathcal{M}$ for Steiner's Problem. Then, of course,

$$L(X, \rho)(\mathcal{M}(N)) \geq L(X, \rho)(\text{SMT for } N). \tag{3.14}$$

We consider the quantity

$$\text{error}(\mathcal{M}) = \max \left\{ \frac{L(\mathcal{M}(N))}{L(\text{SMT for } N)} : N \text{ a finite set} \right\}. \tag{3.15}$$

This measures the quality of an approximation algorithm by its performance ratio. In view of (3.14),

$$\text{error}(\mathcal{M}) \geq 1 \tag{3.16}$$

It is observed that optimization problems which are hard in the sense of computational complexity display different kinds of behaviour in the sense of approximation.[6]
Approximations and heuristics differ in the following sense: For an approximation algorithm, we can estimate the performance ratio with mathematical methods; for a heuristic algorithm, however, we only have experimental results or plausible reasons for the description of the performance ratio.
For a complete discussion of theoretical aspects see [164], [192], [248], and [360].

There are several approximations and heuristics for Steiner's Problem, scattered in the literature. In particular, we have established that it is simple (in any sense) to find an MST. Moreover, the construction of an MST does not need any geometry, it uses only the mutual distances between points.

[5] We will see that such spaces indeed exist.
[6] Performance guarantees must consider the worst-case behavior of an approximation, and they may not reflect how well the approximation actually performs in practice. Thus, performance guarantees should not be the only criterion in evaluating an approximation. Running time, ease of implementation, and empirical analysis are at least as important for the practitioner.

Observation 3.4.1 *Let $\mathcal{M}$ be a method interconnection a set of points in a metric space. Further assume that $\mathcal{M}$ works in each metric space, that is, it only uses the knowledge of ρ and nothing about the geometry of the space. Then $\mathcal{M}$ is the method to find an MST.*

The *proof* is given by 2.3.1 and by a nice argument of Lovász et.al. [255]:

> There is this story about the pessimist and the optimist: They each get a box of assorted candies. The optimist always picks the best; the pessimist eats the worst (to save the better candies for late). So the optimist always eats the best available candy, and the pessimist always eats the worst available candy; and yet, they end up with eating same candies.

□

The performance ratio of an MST is

$$\text{error(MST)} = \frac{1}{m(X,\rho)} \leq 2. \tag{3.17}$$

Of course, for specific spaces, where we have knowledge about the geometry, better algorithms may possible. But, with all these facts in mind, we are only interested in approximations and heuristics satisfying one or both of the following properties:

- The running time of the algorithm is at most the time to compute an MST in this space.
- The error is at most $1/m$, where m is the Steiner ratio of the space.

Note that there are metric spaces for which methods to solve Steiner's Problem are unknown. Then it will be meaningful to approximate an SMT in such spaces by methods which need exponential time.

Provan [302] presented an approximation that transforms Steiner's Problem in the Euclidean plane into Steiner's Problem on a graph such that the relative defect of the lengths of the two SMT's is bounded by a predetermined real number $\epsilon > 0$.[7] Cieslik [80], [81], [83] generalized this result to the following statement: Let N be a finite set in a normed plane, and let ϵ be a positive real number. Then there is an algorithm that finds a tree T interconnecting the points of N and

$$1 \leq \frac{L(T)}{L(\text{SMT for } N)} \leq 1 + \epsilon.$$

[7] The approach is the following: Let N be a given finite set of points and let $\epsilon > 0$ be a real number. Find the values $x[\min], x[\max], y[\min]$ and $y[\max]$ of the minimum and maximum x-coordinate and y-coordinate, respectively, of points in N. Define m as the smallest integer greater than $(8n-12)/\epsilon$. Then divide the segments equally by $x[\min] = x_0 < x_1 < \ldots < x_m = x[\max]$ and $y[\min] = y_0 < y_1 < \ldots < y_m = y[\max]$. Let $V = \{(x_i, y_j) : i, j = 0, \ldots, m\}$ be the lattice of all points with these values as coordinates.
Define the graph $G(\epsilon)$ in the following way:
1. $N \cup (V \cap \text{conv}N)$ is the set of vertices;
2. G is a complete graph;
3. The length of an edge equals the Euclidean distance between the vertices incident to the edge.
Then find an SMT in the graph $G(\epsilon)$ for the set N.

Chapter 4

The Steiner ratio of Banach-Minkowski Spaces

This present chapter concentrates on investigating the Steiner ratio for Banach spaces. The goal is to determine or at least to estimate the Steiner ratio for many different spaces. We distinguish between finite-dimensional Banach spaces, so-called Banach-Minkowski spaces, and general ones.[1]
Our focus on Banach-Minkowski spaces comes from

1. Steiner's Problem in Banach-Minkowski spaces is of great practical interest, see [61], [91], [92]. Hence, it is good investigated, and we have many helpful knowledge about SMT's.

2. In Banach-Minkowski spaces for any finite set of points an SMT always exists; hence, the Steiner ratio is well-defined. In general spaces this does not have to be true, compare [23].

Infinite-dimensional Banach spaces, called Banach-Wiener spaces, will discussed in its own chapter.

4.1 Norms and Balls

Obviously, Steiner's Problem depends essentially on the way how the distances in the plane are determined. In the present paper, at first, we consider finite-dimensional Banach spaces.

I. A_d denotes the d-dimensional affine space with origin o. That means: A_d is a set of points and these points act over a d-dimensional linear space. We identify each

[1] In his book *Geometrie der Zahlen* [271], published in 1896, Minkowski proved a number of results by geometrical arguments, defining the "length" of a vector $v = (t_1, \ldots, t_d)$ to be the quantity $\sqrt[p]{\sum_{i=1}^{d} |t_i|^p}$, where p is a real number with $p \geq 1$.

point with its vector with respect to the origin. In other words, elements of A_d will be called either points when considerations have a geometrical character or vectors when algebraic operations are applied. In this sense, the zero-element o of the linear space is the origin of the affine space.
The dimension of an affine space is given by the dimension of its linear space. A two-dimensional affine space is called a plane. A nonempty subset of an affine space which is itself an affine space is called an affine subspace.

II. The idea of normed spaces is based on the assumption that to each vector of a space can be assigned its "length" or norm, which satisfies some "natural" conditions. A convex and compact body B of the d-dimensional affine space A_d centered in the origin o is called a unit ball and induces a norm $||.|| = ||.||_B$ in the corresponding linear space by the so-called Minkowski functional:

$||v||_B = \inf\{t > 0 : v \in tB\}$ for any v in $A_d \setminus \{o\}$, and

$||o||_B = 0$.

On the other hand, let $||.||$ be a norm in A_d, which means, it is a real-valued function satisfying

(i) positivity: $||v|| \geq 0$ for any v in A_d;

(ii) identity: $||v|| = 0$ if and only if $v = o$;

(iii) homogeneity: $||tv|| = |t| \cdot ||v||$ for any v in A_d and any real t; and

(iv) triangle inequality: $||v + v'|| \leq ||v|| + ||v'||$ for any v, v' in A_d.

Then

$$B = \{v \in A_d : ||v|| \leq 1\} \tag{4.1}$$

is a unit ball in the above sense. It is not hard to see that the correspondences between unit balls B and norms $||.||$ are unique. That means that a norm is completely determined by its unit ball and vice versa. Consequently, a Banach-Minkowski space is uniquely defined by an affine space A_d and a unit ball B. This Banach-Minkowski space is abbreviated as $M_d(B)$. In each case we have the induced norm $||.||_B$.[2]
For example, consider the following unit ball in the d-dimensional space

$$H = \text{conv}([0,1]^d \cup [-1,0]^d). \tag{4.2}$$

Then its norm is

$$||(x_1, \dots, x_d)||_H = \max\{x_i : x_i \geq 0\} - \min\{x_i : x_i \leq 0\}. \tag{4.3}$$

In general, the following fact is easy to see: Let B_i, $i = 1, \dots, s$ be a collection of unit balls. Then $B = B_1 \cap \ldots \cap B_s$ is a unit ball too and

$$||.||_B = \max\{||.||_{B_i} : i = 1, \dots, s\}. \tag{4.4}$$

This is key step to prove

[2] If we drop the assumption that the body B is symmetric with respect to o, the Minkowski functional is not a norm, since $|| - v|| = ||v||$ does not hold in general.

Observation 4.1.1 *Let the unit ball be a convex polytope P. Then the induced norm can be described as follows:*

a) *If P is a $\mathcal{H}$-polytope*

$$P = \bigcap_{j=1}^{s} K(z_j, 1) \tag{4.5}$$

where

$$K(z_j, 1) = \{v \in A_d : (v, z_j) \leq 1\} \tag{4.6}$$

are halfspaces of the A_d. Then the norm derived from P is a piecewise linear function:

$$||v||_P = \max_{j=1,\ldots,s} (v, z_j). \tag{4.7}$$

b) *Let the set of nodes of P as a $\mathcal{V}$-polytope be given*

$$extP = \{\pm v_1, \ldots, \pm v_r\}. \tag{4.8}$$

Then the derived norm can be formulated by

$$||v||_P = \min \left\{ \sum_{i=1}^{r} |\alpha_i| : v = \sum_{i=1}^{r} \alpha_i v_i \right\}. \tag{4.9}$$

Note, that if the unit ball P is the convex hull of finitely many points, that means as $\mathcal{V}$-polytope, then P can be generated as intersections of halfspaces, that means as $\mathcal{H}$-polytope, in an algorithmic way, and vice versa, see Swart [345].[3]

III. A Banach-Minkowski space $M_d(B)$ is a complete metric linear space if we define the metric by

$$\rho(v, v') = ||v - v'||_B. \tag{4.10}$$

Usually, a (finitely- or infinitely-dimensional) linear space which is complete with regard to its given norm is called a Banach space. Essentially, every Banach-Minkowski space is a finite-dimensional Banach space and vice versa.
Note that a metric induced by a norm is translation invariant, which means:

$$\rho(v + w, v' + w) = \rho(v, v'), \tag{4.11}$$

for any v, v', w.
All norms in a finite-dimensional affine space induce the same topology, the well-known topology with coordinate-wise convergence.[4] In other words: In such spaces all norms are topologically equivalent, i.e. there are positive constants c_1 and c_2 such that

$$c_1 \cdot ||.|| \leq |||.||| \leq c_2 \cdot ||.|| \tag{4.12}$$

[3] It is well-known that the class of all polytopes is dense in the class of all bodies. Then the class of all block norms is dense in set of all norms, compare [365].

[4] This is the topology derived from the Euclidean metric.

for the two norms $||.||$ and $|||.|||$.
Conversely, there is exactly one topology that generates a finite-dimensional linear space to a metric linear space satisfying the separating property by Hausdorff, compare [352].

IV. Let $M_d(B)$ and $M_d(B')$ be Banach-Minkowski spaces.
$M_d(B)$ is said to be isometric to $M_d(B')$ if there is a surjective mapping $\Phi : A_d \to A_d$ (called an isometry) which preserves the distances:

$$||\Phi v - \Phi v'||_{B'} = ||v - v'||_B \tag{4.13}$$

for all v, v' in A_d.
It is easy to see that Φ must be an injective mapping. Moreover, a well-known fact given by Mazur and Ulam says that each isometry mapping from a Banach-Minkowski space onto another, such that it maps o on o, is a linear operator and

$$\Phi B = B'. \tag{4.14}$$

In other terms, (4.13) and (4.14) are equivalent.

Lemma 4.1.2 *$M_d(B)$ is isometric to $M_d(B')$ if and only if there is an affine map $\Phi : A_d \to A_d$ with $\Phi B = B'$. Consequently,*

$$||\Phi v||_{\Phi B} = ||v||_B. \tag{4.15}$$

Moreover, the affine map Φ is the isometry itself.

For a proof see [51], [307] or [352].
As an example note that the two parallelograms $B(1) = \text{conv}\{(0, \pm 1), (\pm 1, 0)\}$ and $B(\infty) = \text{conv}\{(\pm 1, \pm 1)\}$ can be transformed with help of

$$\Phi = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{4.16}$$

V. Steiner's Problem looks for a shortest network interconnecting a finite set of points, and thus, in particular for a shortest length of a curve $\mathcal{C}$ joining two points. For our purpose, we regard a geodesic curve as any curve of shortest length: If we parametrize the curve $\mathcal{C}$ by a differentiable map $\gamma : [0, 1] \to \mathbb{R}^d$ we define

$$\text{length of } \mathcal{C} = \int_0^1 ||\dot{\gamma}||\, dt. \tag{4.17}$$

It is not hard to see that among all differentiable curves $\mathcal{C}$ from the point v to the point v' the segment

$$\underline{vv'} = \{tv + (1-t)v' : 0 \le t \le 1\} \tag{4.18}$$

minimizes the length of $\mathcal{C}$.[5]

A unit ball B in an affine space is called strictly convex if one of the following pairwise equivalent properties is fulfilled:

- For any two different points v and v' on the boundary of B, each point $w = tv + (1-t)v', 0 < t < 1$, lies in int$B$.
- No segment is a subset of bd$B$.
- $||v + v'||_B = ||v||_B + ||v'||_B$ for two vectors v and v' implies that v and v' are linearly dependent.

One more property we have in

Lemma 4.1.3 *All segments in a Banach-Minkowski space are shortest curves (in the sense of inner geometry). They are the unique shortest curves if and only if the unit ball is strictly convex.*

Hence, we can define the metric in a Banach-Minkowski space $M_d(B)$ by

$$\rho(v, v') = \frac{2 \cdot ||v - v'||_{B(2)}}{||w - w'||_{B(2)}}, \tag{4.19}$$

where $\underline{ww'}$ is the Euclidean diameter of B parallel to the line through v and v' and $||.||_{B(2)}$ denotes the Euclidean norm.

A function F defined on a convex subset of the affine space is called a convex function if for any two points v and v' and each real number t with $0 \leq t \leq 1$, the following is true

$$F(tv + (1-t)v') \leq tF(v) + (1-t)F(v'). \tag{4.20}$$

A function F is called a strictly convex function if the following is true for any two different points v and v' and each real number t with $0 < t < 1$:

$$F(tv + (1-t)v') < tF(v) + (1-t)F(v'). \tag{4.21}$$

[5] The fourth problem of Hilbert, see [9], [380], is to characterize all geometries in which segments (as the convex hull of two different points) are shortest curves (in the sense of inner geometry). Furthermore, he said

> "One finds that such a geometry really exists and is no other than that which Minkowski constructed in his book "Geometrie der Zahlen", and made the basis of his arithmetical investigations. Minkowski's is therefore also a geometry standing next to the ordinary euclidean geometry; it is essentially characterized by the following stipulations:
> 1. The points which are at equal distances from a fixed point O lie on a convex closed surface of the ordinary euclidean space with O as a center.
> 2. Two segments are said to be equal when one can be carried into the other by a translation of the ordinary euclidean space."

In particular, Hilbert asks for the construction of all these metrics and the study of the individual geometries. Hilbert's comments show that he was not aware of the immense number of these metrics, so the second part of the problem is not a well-known question and has inevitably been replaced by the investigation of special or special classes of interesting geometries.

Lemma 4.1.4 *For a norm $||.||$ in a finite-dimensional affine space the following holds:*

a) $||.||$ *is a convex and thus a continuous function.*

b) $||.||$ *is a strictly convex function if and only if its unit ball $B = \{v \in A_d : ||v|| \leq 1\}$ is a strictly convex set.*

VI. $(.,.)$ denotes the standard inner product, that means for $v = (x_1, \ldots, x_d)$ and $w = (y_1, \ldots, y_d)$ in A_d we define

$$(v, w) = \sum_{i=1}^{d} x_i y_i. \tag{4.22}$$

The Euclidean norm can be derived from the inner product by $||v||_{B(2)} = \sqrt{(v, v)}$. It is easy to see that if for two vectors v and w in a space with an inner product $(.,.)$ it holds the so-called parallelogram law:

$$||v + w||^2 + ||v - w||^2 = 2 \cdot ||v||^2 + 2 \cdot ||w||^2. \tag{4.23}$$

But much harder it is to prove that the converse is also true, [216]: If the so-called polarization identity

$$4 \cdot (v, w) = ||v + w||^2 - || - v + w||^2 \tag{4.24}$$

holds for any vectors v and w, then $(.,.)$ gives an inner product. The importance of this result is to show that being Euclidean is a property of two dimensions, that means

Observation 4.1.5 *A space is Euclidean if and only if every of its two-dimensional subspaces is Euclidean.*

This observation is the starting point of many characterizations.[6] Other characterizations are given by 4.3.5, 4.3.6, [26] and [336].
We assume that the usual Euclidean geometry is well-known. An algebraic approach

[6]For example

Remark 4.1.6 *The following condition defines a Euclidean space: The norm is ptolemaic; that means that for any four points v_1, v_2, v_3 and v_4 the inequality*

$$||v_1 - v_2|| \cdot ||v_3 - v_4|| + ||v_1 - v_4|| \cdot ||v_2 - v_3|| \geq ||v_1 - v_3|| \cdot ||v_2 - v_4||$$

always holds.

For a proof see Day [113].
Melzak's algorithm [266] to find an SMT in the Euclidean plane uses essentially the equality case, that is: A quadrilateral inscribed in a circle has the property that the product of its diagonals equals the sum of the products of the opposite sides, [182], [283] or some standard textbooks on geometry. Now it is clear that Melzak's approach cannot work in other Banach-Minkowski planes.

to the Euclidean geometry is given by Artzy [14].

VII. The dual norm $||.||_{DB}$ of the norm $||.||_B$ is defined as

$$||v||_{DB} = \max_{w \neq o} \frac{(v,w)}{||w||_B} \tag{4.25}$$

and has the unit ball DB, called the dual unit ball, which can be described as

$$DB = \{w : (v,w) \leq 1 \text{ for all } v \in B\}. \tag{4.26}$$

Immediately, we have that for any two vectors v and w the inequality

$$(v,w) \leq ||v||_{DB} \cdot ||w||_B \tag{4.27}$$

is true, and it is not hard to see that $B \subseteq B'$ holds if and only if $DB' \subseteq DB$.
An example of a pair of non-Euclidean norms dual to each other is

$$||(x_1, \dots, x_d)||_{B(\infty)} = \max\{|x_1|, \dots, |x_d|\} \tag{4.28}$$

and

$$||(x_1, \dots, x_d)||_{DB(\infty)} = ||(x_1, \dots, x_d)||_{B(1)} = |x_1| + \dots + |x_d|, \tag{4.29}$$

whereby $B(\infty)$ is a hypercube and $B(1)$ is a cross-polytope.

VIII. Particularly, we are interested in finite-dimensional spaces with p-norm, defined in the following way: Let A_d be the d-dimensional affine space. For the point $v = (x_1, ..., x_d)$ we define the norm by

$$||v||_{B(p)} = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p},$$

where $1 \leq p < \infty$ is a real number. If p runs to infinity, we get the maximum norm

$$||v||_{B(\infty)} = \max\{|x_i| : 0 \leq i \leq d\}.$$

In each case we obtain a Banach-Minkowski space shortly written by $\mathcal{L}_p^d$.
Note that the p-norm satisfy the following monotonicity properties[7]:

Lemma 4.1.7 *If $1 \leq p \leq q \leq \infty$ then for all v in the d-dimensional space*

$$\begin{aligned} ||v||_{B(p)} &\geq ||v||_{B(q)} \quad \textit{and} & (4.30)\\ d^{1/q}||v||_{B(p)} &\leq d^{1/p}||v||_{B(q)}. & (4.31) \end{aligned}$$

In particular,

$$\frac{1}{d} \cdot ||v||_{B(1)} \leq \frac{1}{\sqrt{d}} \cdot ||v||_{B(2)} \leq ||v||_{B(\infty)} \leq ||v||_{B(2)} \leq ||v||_{B(1)}. \tag{4.32}$$

[7]Proof by direct calculations and mean value inequalities.

$\mathcal{L}_1^d$ and $\mathcal{L}_\infty^d$ are normed by a cross-polytope and a cube, respectively. For $1 < p < \infty$ the space $\mathcal{L}_p^d$ is strictly convex. The spaces $\mathcal{L}_p^d$ and $\mathcal{L}_q^d$ with $1/p + 1/q = 1$ are dual, also for the values $p = 1$ and $q = \infty$.[8] The space $\mathcal{L}_2^d$ is self-dual.

IX. For more facts about the structure and geometry of Banach-Minkowski spaces see Martini et al. [260], Schäffer [320] and Thompson [352].

4.2 Steiner's Problem and SMT's

A graph $G = (V, E)$ with the set V of vertices and the set E of edges is embedded in a Banach-Minkowski space $M_d(B)$ normed by $||.||_B$ in the sense that

- V is a finite set of points in the space;
- Each edge $\underline{vv'} \in E$ is a segment $\{tv + (1-t)v' : 0 \leq t \leq 1\}$ (or another geodesic curve) interconnecting $v, v' \in V$; and
- The length of G is defined by

$$L(G) = L(B)(G) = \sum_{\underline{vv'} \in E} ||v - v'||_B.$$

Now, **Steiner's Problem of Minimal Trees** is the following:

Given: A finite set N of points in the Banach-Minkowski space $M_d(B)$.

Find: A connected graph $G = (V, E)$ embedded in the space such that
- $N \subseteq V$ and
- $L(G)$ is as minimal as possible.

A solution of Steiner's Problem is called a Steiner Minimal Tree (SMT) for N in the space.
That there for any finite set of points an SMT always exists is not obvious.[9]

The vertices in the set $V \setminus N$ are called Steiner points. We may assume that the degree of each Steiner point is at least three; and the number of Steiner points is limited: $|V \setminus N| \leq |N| - 2$.

[8] In the following sense $\mathcal{L}_1^d$ and $\mathcal{L}_\infty^d$ are the "extreme cases" of Banach-Minkowski spaces.

Observation 4.1.8 *(Auerbach) For any Banach-Minkowski space $M_d(B)$ there exists a invertible linear mapping Φ such that for all $v \in A_d$,*

$$||\Phi v||_{B(\infty)} \leq ||v||_B \leq ||\Phi v||_{B(1)}. \tag{4.33}$$

In geometric terms: A basic may be chosen in any Banach-Minkowski space such that the cross-polytope is contained in the unit ball of the space which in turn is contained in the hypercube.

[9] Particularly, for finite-dimensional spaces it is proved in [83]. For Banach spaces which are not finite-dimensional this question is not easy to answer, and will be discussed at in its own chapter.

In Banach-Minkowski spaces the condition of length-minimality forces that the degrees of the vertices are bounded from above; we quote results about upper bounds of these degrees, depending only on the space $M_d(B)$. The following table gives some examples of known values for the maximum degree, compare [342].

unit ball	maximum degree of a vertex
Euclidean	3
cube	2^d
cross-polytope	$2d$

Because of its specific interest, we discuss the Euclidean case, where the degree of the Steiner points is independently from the dimension.

Example 4.2.1 *The degree of a vertex in an SMT is at most three and the degree of a Steiner point in Euclidean spaces (of any dimension) equals three.*

Proof. In view of 2.4.1 we have for a set $N = \{v_1, \ldots, v_n\}$

$$\sum_{i<j} \frac{(u_i, u_j)}{||u_i|| \cdot ||u_j||} = -\frac{n}{2}, \tag{4.34}$$

where $u_i = q - v_i$ denotes vector given by the the Torricelli point q and the given points v_i, $i = 1, \ldots, n$. The equation (4.34) holds true in d dimensions. Hence,

$$-\frac{n}{2} \leq \left(-\frac{1}{2}\right) \cdot \binom{n}{2},$$

that is, an inequality which is satisfied only for $n \leq 3$.

□

Let $z(d)$ be the maximum possible degree of a vertex and $s(d)$ be the maximum possible degree of a Steiner point in an SMT in a d-dimensional normed space, respectively. Cieslik [72], [83] has shown that $z(d)$ really exists; namely he proved $z(d) \leq 3^d - 1$ and conjectured

Conjecture 4.2.2

$$z(d) \leq 2 \cdot (2^d - 1). \tag{4.35}$$

The conjecture was well-supported, [78], [112], but in full generality not true, [344].

Assuming that the unit ball is a polytope. Since any edge joining two vertices in an SMT can be replaced by a piecewise linear path consisting of segments parallel to the vectors pointing to the nodes of the unit ball, we obtain

Theorem 4.2.3 *(Du, Hwang [130]) Let the unit ball B be a (convex) polytope with z nodes. Then the degree of each vertex of an SMT in $M_d(B)$ is at most z.*

It is not hard to see that $s(d) \leq z(d)$, and Morgan [277], [279] conjectured

Conjecture 4.2.4

$$s(d) \leq 2^d. \tag{4.36}$$

The bound 2^d is achieved by the space $M_d(B(\infty))$, since the star joining the origin to the 2^d vertices of the unit ball is an SMT. Thus the upper bound (4.36), if the conjecture is true, would be the best possible.
Swanepoel [343], [344] gives the previously best known upper bound

$$z(d) \leq O(2^d d^2 \log d). \tag{4.37}$$

Both conjectures (4.35) and (4.36) are true in the planar case: $z(2) = 6$, Cieslik [74] and $s(2) = 4$, Swanepoel [342]. This also shows that $s(d) < z(d)$ is possible, but Swanepoel [344] showed that the plane normed by an affinely regular hexagon is the only case of a two-dimensional space in which both numbers differ. In all other cases $s(2) = z(2) \in \{3, 4\}$.[10] Moreover, Cieslik [74], [83] shows that for any finite set of points there exists an SMT with all vertices of degree four.
The two-dimensional methods are very special and offer no hope for generalizations to higher dimensions. But, further investigations for determining these quantities more exactly are necessary, since these numbers have a deep influence in creating fast approximations for shortest networks, compare [93]. For instance,

Remark 4.2.5 *(Lawlor, Morgan [243]) Let $M_d(B)$ be a Banach-Minkowski space normed by a smooth unit ball. Then the degree of a vertex in an SMT is at most $d + 1$.*

In particular in a Banach-Minkowski plane with a smooth unit ball any Steiner point in an SMT has degree three. In other terms, $s = 3$ does not characterize Euclidean spaces.

A similar quantity is the maximum possible degree of a vertex in an MST, see [90]. Here $3^d - 1$ is a sharp upper bound, achieved by the hypercube as unit ball, which creates the maximum norm [181], [172].[11] In particular, in an MST for a finite set of

[10] $s(2) \leq 4$ gives an approach to reduce Steiner's Problem in Banach-Minkowski planes to simpler ones [99]: We have only to consider a full Steiner tree $T = (V, E)$ for $N = \{v_1, \ldots, v_n\}$. That means, let $V = \{v_1, \ldots, v_{2n-2}\}$, whereby $g(v_i) = 1$ for $i = 1, \ldots, n$ and $g(v_i) = 3$ for $i = n+1, \ldots, 2n-2$. Let $A(T) = (a_{ij})_{i,j=1,\ldots,2n-2}$ be the adjacency matrix of T. Then it is only necessary to minimize the function

$$\begin{aligned} S_B(T) &= S_B(v_{n+1}, \ldots, v_{2n-2}) \\ &:= \sum_{i=1}^{n} \sum_{j=n+1}^{2n-2} a_{ij} ||v_i - v_j||_B + \sum_{i=n+1}^{2n-3} \sum_{j=i+1}^{2n-2} a_{ij} ||v_i - v_j||_B. \end{aligned} \tag{4.38}$$

[11] The problem to determine this so-called kissing numbers was formulated in the context of densest packings of convex bodies, starting with Kepler. Hilbert listed the problem as a part of the 18'th open questions. Compare [108] or [386] for the mathematics and [347] for the history of packing problems.

points in the Euclidean plane, each vertex has degree at most six, and there exists an MST with maximal degree five, [90].

4.3 The Steiner Ratio of specific Spaces

$$m(M_d(B)) = m_d(B) := \inf\left\{\frac{L(B)(\text{SMT for } N)}{L(B)(\text{MST for } N)} : N \subseteq M_d(B) \text{ a finite set}\right\}, \quad (4.39)$$

is called the Steiner ratio of the Banach-Minkowski space $M_d(B)$. For the space $\mathcal{L}_p^d$ it is briefly written by $m(d,p)$.

I. In the d-dimensional affine space A_d, the unit ball $B(1)$ is the convex hull of

$$N = \{\pm(0,...,0,1,0,...,0) : \text{the i'th component is equal to } 1, i = 1,\ldots,d\}. \quad (4.40)$$

The set N contains $2d$ points. The rectilinear distance of any two different points in N equals 2. Hence, an MST for N has the length $2(2d-1)$. Conversely, an SMT for N with the Steiner point $o = (0,...,0)$ has the length $2d$:

$$\mu(N) \leq \frac{2d}{2(2d-1)} = \frac{d}{2d-1}. \quad (4.41)$$

Theorem 4.3.1 *For the Steiner ratio of spaces with rectilinear norm*

$$m(d,1) \leq \frac{d}{2d-1}. \quad (4.42)$$

It is conjectured that this bound is the best possible one:

Conjecture 4.3.2 *(Graham and Hwang [171]) In (4.42) always equality holds.*

4.3.2 is true in the planar case, which means that a rectilinear MST is never longer than three-halves the length of the rectilinear SMT: $m(2,1) = 2/3$, shown by Hwang [199], but the methods do not seem to be applicable to prove the conjecture in the higher-dimensional case.[12]
Since $d/(2d-1)$ runs to $1/2$ when d goes to infinity, we find, together with 3.2.1,

Corollary 4.3.3 *The lower bound* $1/2$ *is the best possible for the Steiner ratio over the class of all Banach-Minkowski spaces.*

II. Let $M_d(B)$ and $M_d(B')$ be Banach-Minkowski spaces. A surjective mapping $\Phi : M_d(B) \to M_d(B')$ with the property

$$||\Phi v - \Phi v'||_{B'} = ||v - v'||_B \quad (4.43)$$

for all v, v' in A_d is called an isometry. It is easy to see that Φ must be an injective mapping. In view of 4.1.2,

[12] The difficulty for extending Hwang's approach is due to the lack of knowledge on full Steiner trees in $M_d(B(1))$. For $d = 2$ all Steiner points lie on a path of the tree. A similar fact for higher dimensions is unknown.

Lemma 4.3.4 *If there exists an isometry between the Banach-Minkowski spaces $M_d(B)$ and $M_d(B')$, then*

$$m_d(B) = m_d(B'). \tag{4.44}$$

This relatively simple fact has a lot of interesting consequences:

- Every parallelogram B in the affine plane A_2 is the image of the "square" $B(1)$ under an affine transformation. Consequently, it induces the same Steiner ratio, namely the Steiner ratio of the plane with rectilinear norm and the plane with maximum norm:

  $$m_2(B) = m(\mathcal{L}_1^2) = m(\mathcal{L}_\infty^2). \tag{4.45}$$

  Whereas in the plane a hypercube and a cross-polytope are "squares", these bodies in higher-dimensional spaces are different, that means that there does not exist an affine map which transforms one into the other. That is, $\mathcal{L}_1^d$ is not isometric to $\mathcal{L}_\infty^d$, $d \geq 3$.

- The Euclidean ball is the set $B(2) = \{x : (x, x) \leq 1\}$. Then for an affine transformation Φ, the set $\Phi B(2) = B^{(e)}$ is called an ellipsoid. All ellipsoids B in the affine space A_d induce the same Steiner ratio, namely the Steiner ratio of the Euclidean space:

  $$m_d(B) = m_d(B(2)) = m(\mathcal{L}_2^d). \tag{4.46}$$

- Let B and B' be two unit balls in A_d. B and B' are called similar if $B = cB'$ for some positive real number c. The lemma implies that the Steiner ratios are equal:

  $$m_d(B) = m_d(B'). \tag{4.47}$$

III. Let $M_d(B)$ be a d-dimensional Banach-Minkowski space, and let $A_{d'}$ be a d'-dimensional affine subspace ($d' \leq d$) with $o \in A_{d'}$. Clearly, the intersection $B \cap A_{d'}$ can be considered as the unit ball of the space $A_{d'}$. This means that $M_{d'}(B \cap A_{d'})$ is a (Banach-Minkowski) subspace of $M_d(B)$.

Let v and v' be two different points in $A_{d'}$. Then the line through v and v' lies completely in $A_{d'}$, and in view of 4.1.3 and (4.19), we see that the distance between the points v and v' is preserved:

$$||v - v'||_B = ||v - v'||_{B \cap A_{d'}}. \tag{4.48}$$

Kruskal's method 2.3.1, which finds an MST, uses only the mutual distances between the points. Hence,

$$L(B)(\text{MST for } N) = L(B \cap A_{d'})(\text{MST for } N)$$

for any finite set N of points in $M_{d'}(B \cap A_{d'})$. On the other hand, it is possible that an SMT for N in the space $M_d(B)$ is shorter than in the subspace $M_{d'}(B \cap A_{d'})$.[13] Consequently,

$$L(B)(\text{SMT for } N) \leq L(B \cap A_{d'})(\text{SMT for } N)$$

for any finite set N of points in $M_{d'}(B \cap A_{d'})$. Then we have

Theorem 4.3.7 *Let $M_{d'}(B')$ be a (Banach-Minkowski) subspace of $M_d(B)$. Then*

$$m_{d'}(B') \geq m_d(B).$$

It is an interesting question, whether in the case of strict subspaces the inequality is also strict. Furthermore, an open problem is to bound the ratio of the space by a quantity depending on the ratio of the subspace.[14]

[13]The reason is that the Torricelli point of a set N might not lie in the affine hull of N. We got an example in the following observation in the three-dimensional space: Consider the function

$$F_{N,B(p)}(w) = \sum_{i=1}^{3} ||v_i - w||_{B(p)}.$$

Let N be the set of the three points $v_1 = (1,0,0), v_2 = (0,1,0)$ and $v_3 = (0,0,1)$ in $M_3(B(p))$. A Steiner point for v_1, v_2, v_3 must minimize $F_{N,B(p)}(\cdot)$.
Suppose that the Torricelli point of these points lies in the plane determined by v_1, v_2 and v_3, that is aff$N = \{(x,y,z) : x+y+z = 1\}$. The strict convexity of the p-norm has the consequence that there is a unique minimum in this plane; the symmetry of v_1, v_2 and v_3 implies that $v_0 = (1/3, 1/3, 1/3)$ is this point. On the other hand, since the function $F_{N,B(p)}(x,y,z)$ attains its minimum value at v_0, the following must be true as well:

$$\frac{\partial F_{N,B(p)}}{\partial x}|_{v=v_0} = \frac{\partial F_{N,B(p)}}{\partial y}|_{v=v_0} = \frac{\partial F_{N,B(p)}}{\partial z}|_{v=v_0} = 0,$$

that is

$$-\left(\frac{2}{3}\right)^{p-1} + 2\left(\frac{1}{3}\right)^{p-1} = 0.$$

This implies that $p = 2$. Hence, for p different from 2, the Torricelli point does not lie in the plane spanned by N. Moreover in the planar case for any finite set of points there is a Torricelli point in conv$N$, compare [70]. But,

Observation 4.3.5 *(Durier [141]) Let $M_d(B)$ be a Banach-Minkowski space, where the dimension d is greater than 2. Suppose that a Torricelli point for all subsets N with three or four elements is contained in the affine hull of N. Then $M_d(B)$ is an inner product (essentially an Euclidean) space.*

And, improved

Observation 4.3.6 *(Benitez et al. [25]) Let $M_d(B)$ be a Banach-Minkowski space, where the dimension d is greater than 2. Suppose that the set of all Torricelli points for all sets N with three elements intersects the convex hull of N. Then $M_d(B)$ is an inner product space.*

[14]Is the consideration of projection constants, compare Grünbaum [175], helpful?

4.4 The Banach-Mazur Distance

In (4.12) we said that any two norms of a finite-dimensional affine space are equivalent. More exactly: Let $\underline{B}_d$ denote the class of all unit balls of the d-dimensional affine space A_d. Since B and B' in $\underline{B}_d$ are compact bodies, there are positive real numbers c and c' such that

$$\frac{1}{c} \cdot B \subseteq B' \subseteq \frac{1}{c'} \cdot B. \tag{4.49}$$

Hence, for any v in A_d

$$c \cdot ||v||_B \geq ||v||_{B'} \geq c' \cdot ||v||_B \tag{4.50}$$

Let N be a finite set of points in A_d. Assume that $T = (V, E)$ is a tree for N, that means $N \subseteq V$, particularly, an SMT or an MST for N. Then

$$\begin{aligned} c \cdot L(B)(T) &= c \cdot \sum_{\underline{vv'} \in E} ||v - v'||_B = \sum_{\underline{vv'} \in E} c \cdot ||v - v'||_B \\ &\geq \sum_{\underline{vv'} \in E} ||v - v'||_{B'} = L(B')(T), \end{aligned}$$

and similarly, $L(B')(T) \geq c' \cdot L(B)(T)$. Consequently, we have

$$c \cdot L(B)(T) \geq L(B')(T) \geq c' \cdot L(B)(T) \tag{4.51}$$

for each tree T interconnecting a finite set of points in A_d. With these facts in mind,

Theorem 4.4.1 *Let B and B' be unit balls in A_d with*

$$\frac{1}{c} \cdot B \subseteq B' \subseteq \frac{1}{c'} \cdot B,$$

where $c \geq c'$ are positive real numbers; or equivalently,

$$c \cdot ||v||_B \geq ||v||_{B'} \geq c' \cdot ||v||_B,$$

for all $v \in A_d$. Then

$$\frac{c}{c'} \cdot m_d(B) \geq m_d(B') \geq \frac{c'}{c} \cdot m_d(B).$$

The Banach-Mazur distance is a distance measure for two Banach-Minkowski spaces. In a first view, we introduce this distance function between classes of Banach-Minkowski spaces in the following way: $\underline{B}_d$ denotes the class of all unit balls in A_d, and let $[\underline{B}_d]$ be the space of classes of isometries for $\underline{B}_d$. Let $j : \underline{B}_d \to [\underline{B}_d]$ be the canonical mapping. Then the Banach-Mazur distance on $[\underline{B}_d]$ is defined as

$$\begin{aligned} \Delta([B], [B']) &= \ln \inf\{h \geq 1 : \textit{there are } B_1 \in j^{-1}([B]) \textit{ and} \\ &\qquad B_2 \in j^{-1}([B']) \textit{ such that } B_1 \subseteq B_2 \subseteq hB_1\} \end{aligned} \tag{4.52}$$

and, equivalently

$$\begin{aligned}\Delta([B],[B']) \quad &= \quad \ln\inf\{h \geq 1 : \textit{there is an isometry } \Phi \textit{ such that} \\ &\qquad B \subseteq \Phi B \subseteq hB\} \end{aligned} \tag{4.53}$$

for $[B], [B']$ in $[\underline{B}_d]$.[15]

Let N be a finite set of points in the affine space A_d, and let $T = (V, E)$ be a shortest tree for N in $M_d(B)$. Consider the Banach-Minkowski space $M_d(B')$. Suppose that $h = \Delta([B], [B'])$. Then

$$B \subseteq \Phi B' \subseteq \exp(h) \cdot B$$

where Φ is a suitably chosen isometry. With help of (4.51), we find that

$$L(B)(T) \geq L(\Phi B')(T) \geq \exp(-h) \cdot L(B)(T).$$

On the other hand, 4.3.4 says that

$$L(\Phi B')(T) = L(B')(\Phi T),$$

where $\Phi T = (\Phi V, \Phi E)$. Consequently, we have that the Steiner ratio as a map from the set of all classes of unit balls into the real numbers is continuous. More exactly,

Theorem 4.4.2 *(Cieslik [92]) Let B and B' be unit balls in the d-dimensional affine space A_d. Then*

$$e^{\Delta([B],[B'])} \cdot m_d(B) \geq m_d(B') \geq e^{-\Delta([B],[B'])} \cdot m_d(B).$$

Proof. There is a sequence $\{h_k\}_{k=1,\ldots\infty}$ with $h_k \to \exp(\Delta([B], [B']))$, where for each number k there are unit balls $B_{1,k} \in j^{-1}([B])$ and $B_{2,k} \in j^{-1}([B'])$ with

$$B_{1,k} \subseteq B_{2,k} \subseteq h_k B_{1,k}.$$

In view of 4.4.1, this implies the inequalities

$$h_k \cdot m_d(B_{1,k}) \geq m_d(B_{2,k}) \geq \frac{m_d(B_{1,k})}{h_k}.$$

Together with 4.3.4, we obtain

$$h_k \cdot m_d(B) \geq m_d(B') \geq \frac{m_d(B)}{h_k}.$$

Hence, if k tends to infinity, one has the assertion.

□

[15] The Banach-Mazur distance Δ is a pseudometric, but not a metric, since $\Delta([B], [B']) = 0$ implies only that B and B' are isometrically.

4.5 The Banach-Mazur Compactum

A compactness argument shows that the infimum in the definition of the Banach-Mazur distance is attained, and we get a function which measures the distance between spaces. In the Banach space context, that means that the definition is phrased in a way that is also applicable to infinite-dimensional spaces; this distance function for two isomorphic normed spaces X and Y is usually defined as

$$\Delta(X,Y) = \log\inf\{||\Phi|| \cdot ||\Phi^{-1}||\}, \tag{4.54}$$

where the infimum is taken over all linear transformations Φ of X onto Y.
The space $([\underline{B}_d], \Delta)$ is a compact metric space, called the Banach-Mazur compactum, sometimes written $BM(d)$. Our first interest is to estimate the diameter of $BM(d)$.

Remark 4.5.1 *(Glushkin, in [291]) There is a positive number a, independent from the dimension d, such that $(\ln d - a)$ is a lower bound for the diameter of the d-dimensional Banach-Mazur compactum BM(d) = $([\underline{B}_d], \Delta)$. Consequently,*

$$\ln d - a \leq \textit{diameter of } BM(d) \leq \ln d.$$

In other terms,

$$\textit{diameter of } BM(d) = \Theta(\ln d).$$

4.5.1 implies that the radius of BM(d) is $\Theta(\ln\sqrt{d}) = \Theta(\ln d)$. Then 4.4.2 has the consequence that $O(d^c) \cdot m_d(B) \geq m_d(B') \geq m_d(B)/\Omega(d^{c'})$, where $c, c' > 0$.[16]
For p-norms we have the Banach-Mazur distance in an explicit formula.

Remark 4.5.3 *(Gurari, Kadec, Macaev [177]) Let p and p' be real numbers with $1 \leq r, q \leq 2$ or $2 \leq r, q \leq \infty$. Then the following is true in the d-dimensional Banach-Mazur compactum:*

$$\Delta([B(r)], [B(q)]) = \left|\frac{1}{r} - \frac{1}{q}\right| \cdot \ln d.$$

Two applications:

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[16] All these facts are geometrically known as

Theorem 4.5.2 *(John's lemma [215]) Let K be a convex and compact body with non-empty interior in the d-dimensional space. Then there exists an ellipsoid $K^{(e)}$ such that*

$$K^{(e)} \subseteq K \subseteq d \cdot K^{(e)}.$$

If K is symmetric about the origin, we have the improved approximation

$$K^{(e)} \subseteq K \subseteq \sqrt{d} \cdot K^{(e)}.$$

To prove 4.5.2 consider an ellipsoid of maximal volume included in K. By compactness the existence of such an ellipsoid is obvious but it can be also shown its unicity. By duality, this clearly implies the existence of a unique ellipsoid of minimum volume containing K.

I. The unit ball

$$\begin{aligned} B^c &= \{(x,y) : x^2+y^2 \le 2,\ x^2/2+y^2 \le 4/3,\ -1 \le y \le 1\} \\ &= \{(x,y) : x^2+y^2 \le 2\} \cap \{(x,y) : x^2/2+y^2 \le 4/3\} \\ &\quad \cap \{(x,y) : -2 \le x \le 2, -1 \le y \le 1\} \end{aligned}$$

is a center of the two-dimensional Banach-Mazur compactum, see [15].[17] We have

$$||(x,y)||_{B^c} = \begin{cases} |y| & : \ |y| \ge \sqrt{\frac{3}{2}}|x| \\ \sqrt{\frac{3}{8}\cdot x^2 + \frac{3}{4}\cdot y^2} & : \ \sqrt{\frac{3}{2}}|x| \ge |y| \ge \frac{|x|}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\sqrt{x^2+y^2} & : \ |y| \le \frac{|x|}{\sqrt{2}} \end{cases}$$

Theorem 4.5.4 *Assuming that the Steiner ratio of the Euclidean plane equals $\sqrt{3}/2$. Let B^c be the unit ball in the plane as defined above. Then,*

$$0.75 = \frac{3}{4} \le m_2(B^c) \le 1 - \frac{1}{\sqrt{27}} = 0.80754\ldots.$$

Proof. To find the upper bound, consider the four-point set

$$N = \left\{ \left(\pm\frac{2}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}} \right) \right\}. \tag{4.55}$$

It is easy to calculate that $N \subset \mathrm{bd}B^c$ forms the nodes of a "square" in $M_2(B^c)$ with side-length $\sqrt{8/3}$. Hence, an MST for N has the length $\sqrt{24}$. If we add the Steiner points $\pm(2/\sqrt{3} - 2/3, 0)$, we find a tree of length $\sqrt{24} - \sqrt{8}/3$. This implies the assertion for the upper bound.
For the lower bound, we use that $B^e = \{(x,y) : x^2/2 + y^2 \le 4/3\}$ is an ellipse which contains B^c. It is not hard to see that

$$\Delta([B^c],[B^e]) = \ln\frac{2}{\sqrt{3}}. \tag{4.56}$$

Hence, in view 4.4.2,

$$m_2(B^c) \ge \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3}{4}. \tag{4.57}$$

□

II. This example of a Banach-Minkowski space comes from the underground mining industry, compare Brazil et al. [44], [46]. Here, the problem is to design a network of tunnels interconnecting a set of given underground locations where ore is concentrated. Because of limitations in the trucks used to haul the ore, the tunnels are not

[17] Note that there are several other centers of $BM(2)$.

allowed to be too steep. Consequently, let's say we constrain the gradient of each edge to be at most g. Apart from this constraint, the distance is Euclidean:

$$||(x,y,z)||_{B^{\text{mine-g}}} = \begin{cases} \sqrt{x^2+y^2+z^2} & : \quad |z| \le g\sqrt{x^2+y^2} \\ \sqrt{1+\frac{1}{g^2}} \cdot |z| & : \quad \text{otherwise} \end{cases}$$

The unit ball $B^{\text{mine-g}}$ is the Euclidean ball with the north and the south poles are sliced off. This implies

$$\sqrt{\frac{g^2}{g^2+1}} \cdot B(2) \subseteq B^{\text{mine-g}} \subseteq B(2). \tag{4.58}$$

In view of 4.4.1

Theorem 4.5.5

$$\sqrt{\frac{g^2+1}{g^2}} \cdot m_3(B(2)) \ge m_3(B^{mine\text{-}g}) \ge \sqrt{\frac{g^2}{g^2+1}} \cdot m_3(B(2)). \tag{4.59}$$

Of course, it is of interest to discuss the configurations giving the Steiner ratio. Prendergast et al. [298] show that there are an infinite number of triangles with differing orientations which achieve it. Consequently here, and in [299], the quantity $m_3^3(B^{\text{mine-g}})$ for specific values of g is discussed.

4.6 The Euclidean Plane

Consider three points v_1, v_2, v_3 which form the nodes of an equilateral triangle of unit side length in the Euclidean plane. An MST for these points has length 2. An SMT has exactly one Steiner point w which is located such that two edges which are incident to w meet at an angle of 120^o. Therefore, with the help of a simple calculation, using the cosine law, we find that the length of the SMT is $3 \cdot \sqrt{1/3} = \sqrt{3}$. So we have an upper bound for the Steiner ratio of the Euclidean plane:

$$m_2(B(2)) \le \frac{\sqrt{3}}{2} = 0.86602\ldots. \tag{4.60}$$

The central question for many further considerations is the following: Is the inequality in (4.60) sharp? That means: Is there a finite set N of points in the Euclidean plane with $\mu(N) < \sqrt{3}/2$ or not? Until today such set was not found, but, of course, this is not a proof of non-existence.
To investigate this question more systematically, let us consider the history of the determination of the Euclidean Steiner ratio: A long-standing conjecture, given by Gilbert and Pollak in 1968, asserts that in the above inequality (4.60), equality holds:

Conjecture 4.6.1 *For the Euclidean plane the following is true:*

$$m(2,2) = m_2(B(2)) = \frac{\sqrt{3}}{2} = 0.86602\ldots. \tag{4.61}$$

This was the most important conjecture in the area of Steiner's Problem in the following years. Many people have tried to show this: Pollak [297] and Du, Yao and Hwang [120] have shown that the conjecture is valid for sets N consisting of $n = 4$ points; Booth [39], and Du, Hwang and Yao [123], and Friedel and Widmayer [155] extended this result to the case $n = 5$, and Rubinstein and Thomas [309] have done the same for the case $n = 6$.
Rubinstein, Thomas [310] showed that the conjecture is valid for points on a circle. On the other hand, many attempts have been made to estimate the Steiner ratio for the Euclidean plane from below:

$m \geq 1/\sqrt{3}$	$= 0.57735\ldots$	Graham, Hwang, 1976, [171]
$m \geq \sqrt{2\sqrt{3} + 2 - (7 + 2\sqrt{3})}$	$= 0.74309\ldots$	Chung, Hwang, 1978, [67]
$m \geq 4/5$	$= 0.8$	Du, Hwang, 1983, [121]
m	$\geq 0.82416\ldots$	Chung, Graham, 1985, [66]

Finally, in 1990, Du and Hwang [125], [129] created many new methods and said that they succeeded in proving the Gilbert-Pollak conjecture completely.[18] But it seems that the proof by Du and Hwang is not correct. Innami et al. [204] describe a mistake, which was a key step of the former proof. That means, the Gilbert-Pollak conjecture is still open. For some background information and references see Ivanov, Tuzhilin [213]. Nevertheless, it would be a surprise if 4.6.1 fails.

4.7 The Steiner Ratio of $\mathcal{L}_p^2$

In this section we will determine upper bounds for the Steiner ratio of $\mathcal{L}_p$-planes, $1 \leq p \leq \infty$, abbreviated by $m(2,p)$. That is $m(2,p) = m_2(B(p)) = m(\mathcal{L}_p^2)$.

I. With usual methods of calculus we can verify that for $v = (x, y)$ the function

$$f(p) = ||v||_{B(p)} = (|x|^p + |y|^p)^{1/p}$$

is decreasing, but the function

$$g(p) = \left(\frac{|x|^p + |y|^p}{2}\right)^{1/p}$$

is increasing. Therefore, for all $1 \leq p \leq q \leq \infty$.

$$||v||_{B(q)} \leq ||v||_{B(p)} \leq 2^{\frac{1}{p} - \frac{1}{q}} \cdot ||v||_{B(q)}, \tag{4.62}$$

for all $v \in A_d$. In particular,

$$||v||_{B(\infty)} \leq ||v||_{B(p)} \leq 2^{\frac{1}{p}} \cdot ||v||_{B(\infty)}.$$

Combining (4.62) with 4.4.1 gives

[18] This mathematical fact appeared in The New York Times, October 30, 1990 under the title "Solution to Old Puzzle: How Short a Shortcut?"

Theorem 4.7.1 *(Liu, Du [252]) Suppose that $1 \le p \le q \le \infty$. Then*

$$2^{\frac{1}{p}-\frac{1}{q}} \cdot m(2,q) \ge m(2,p) \ge \frac{1}{2^{\frac{1}{p}-\frac{1}{q}}} \cdot m(2,q).$$

Together with 4.4.2

Theorem 4.7.2 *(Cieslik [77], [89], [92]) The following inequalities are true for the Steiner ratio of the $\mathcal{L}_p$-planes $M_2(B(p))$:*

$$m(2,p) \le \frac{4}{3} \cdot 2^{-1/p}$$

if $p \le 2$, and

$$m(2,p) \ge \begin{cases} \frac{1}{3} \cdot 2^{1/p} & : 1 \le p \le \frac{ln\ 16}{ln\ 13.5} = 1.06527\ldots \\ \frac{\sqrt{6}}{2} \cdot 2^{-1/p} & : \frac{ln\ 16}{ln\ 13.5} \le p \le 2. \end{cases}$$

We can find bounds for $m(2,p)$, $p \ge 2$, if we replace p by $p/(p-1)$ on the right hand side.

The theorem implies: $m(2,p) \ge \sqrt[4]{1/6} = 0.63894\ldots$ for each number p. But we will find a better bound shortly.

Corollary 4.7.3 *Let p be a real number with $\frac{ln\ 16}{ln\ 13.5} \le p \le 2$. Then*

$$m(2,p) \approx c \cdot 2^{-1/p}, \tag{4.63}$$

where $1.224\ldots \le c \le 1.333\ldots$.

II. Du and Liu determined an upper bound for the Steiner ratio of $\mathcal{L}_p$-planes, using direct calculations of the ratio between the length of SMT's and the length of MST's for sets with three elements:

Theorem 4.7.4 *(Liu, Du [252]) The following is true for the Steiner ratio of the $\mathcal{L}_p$-planes $M_2(B(p))$:*

$$m(2,p) \le \frac{(2^p-1)^{1/p} + (2^q-1)^{1/q}}{4}, \tag{4.64}$$

where $1 < p < \infty$ and q is the conjugate of p; that means $\frac{1}{p} + \frac{1}{q} = 1$.

The *proof* considers the points $u = (1/2, a_p)$, $v = (1,0)$ and $w = (0,0)$ with $a_p = (1-2^{-p})^{1/p}$. We may assume that other triangles give better bounds. Now, we will consider another triangle which has a side parallel to the line $\{(x,x) : x \in I\!R\}$. Let $1 < p < \infty$ and $u = (0,1)$, $v = (1,0)$ and $w = (x_p, x_p)$. We wish that the triangle spanned by u,v and w is equilateral and, additionally, x_p lies between 1 and 2. Hence, x_p is a zero of the function f where

$$f(x) = x^p + (x-1)^p - 2.$$

Of course, f is a strictly monotonically increasing and continuous function. Hence, $f(1) = -1$ and $f(2) = 2^p - 1 > 0$ imply the existence and uniqueness of x_p. Then,

$$L(\text{MST for } \{u, v, w\}) = 2 \cdot 2^{1/p}.$$

□

A further discussion of 4.7.4 gives

Theorem 4.7.5 *(Liu, Du [252]) For $1 \le p \le \infty$, it holds*

$$m(2,p) \le \frac{\sqrt{3}}{2}. \tag{4.65}$$

The equality in this inequality is given when we showed that the conjecture 4.6.1 is true. In this case, equality holds if and only if $p = 2$.

Theorem 4.7.6 *(Albrecht [3], Albrecht, Cieslik [6]) Let $1 < p < \infty$, let x_p be a zero of*

$$f(x) = x^p + (x-1)^p - 2,$$

and let z_p be a real minimizing

$$g(z) = 2(z^p + (1-z)^p)^{1/p} + (x_p - z) \cdot 2^{1/p}.$$

Then

$$m(2,p) \le \left(\frac{z_p^p + (1-z_p)^p}{2}\right)^{1/p} + \frac{1}{2}(x_p - z_p). \tag{4.66}$$

This result gives the following estimates for $m(2,p)$ for specific values of p and its conjugated value q:

p	q	4.7.4	(4.66) with p	(4.66) with q
1.1	11	0.782399...	0.775933...	0.775933...
1.2	6	0.809264...	0.797975...	0.797975...
1.3	4.3...	0.829043...	0.816708...	0.816708...
1.4	3.5	0.842759...	0.832320...	0.832320...
1.5	3	0.852049...	0.844625...	0.844625...
1.6	2.6...	0.858207...	0.853640...	0.853640...
1.7	2.428571...	0.862145...	0.859755...	0.859755...
1.8	2.25	0.864491...	0.863518...	0.863518...
1.9	2.1...	0.865681...	0.865460...	0.865460...
2.0	2	0.866025...	0.866025...	0.866025...

Using only three points, 3.2.4 says that we cannot derive a Steiner ratio less than 3/4. Hence, we have to investigate sets with four points to get sharper estimates.
Albrecht [3] found an upper bound for the Steiner ratio, considering the extreme

points of the sets $B(1)$ and $B(\infty)$ in $\mathcal{L}_p^2$. This idea suggests that we consider the four given points $u = (x_p, 0)$, $v = (0, 1)$, $w = (-x_p, 0)$ and $s = (0, -1)$. Let $q_1 = (a_p, b_p)$ and $q_2 = -q_1$ be Steiner points. The tree T contains the edges $\underline{q_1 u}$, $\underline{q_1 v}$, $\underline{q_1 q_2}$, $\underline{q_2 w}$ and $\underline{q_2 s}$, since each Steiner point has a degree of at least three.

Theorem 4.7.7 *(Albrecht [3], Albrecht, Cieslik [6]) The Steiner ratio of $\mathcal{L}_p^2$ is essentially smaller than $\frac{3}{4}$ if $p \leq 1.2$ or if $p \geq 6$.*

Albrecht [3] also remarked that neither construction gives an SMT, that means the bounds are upper bounds and never exact values for the Steiner ratio $m(2, p)$.

4.8 λ-Geometries

I. It is an interesting question to consider planes which are normed by a regular polygon with an even number of corners.
We define the $\lambda-$geometry $M_2(B^{(\lambda)})$ in the following way: The unit ball $B^{(\lambda)}$ is a regular 2λ-gon, $\lambda > 1$, with the x-axis being a diagonal direction.[19]
Motivated by application in engineering, Hanan [187] considered SMT's in the plane $\mathcal{L}_1^2 = M_2(B(1))$, where

$$||(x, y)||_{B^{(2)}} = ||(x, y)||_{B(1)} = |x| + |y|,$$

called the rectilinear norm. This is the distance between two points if one is only allowed to use a sequence of vertical and horizontal lines as geodesic curves.[20] Moreover, Hanan [187] shows that an SMT always exists as a subgraph of of a grid graph, obtaining by constructing horizontal and vertical lines through each given point. This norm is the main distance used in VLSI design.[21] Recently, more orientations have also been considered in this area. For example, if we allow three orientations, each at 60^o with respect to each other, we obtain the norm in which the unit ball is a regular hexagon, that means essentially, up to isometry

$$||(x, y)||_{B^{(3)}} = \max\{|x|, |y|, |x - y|\}.$$

With four orientations, each at 45^o, we obtain the norm in which the unit ball B is a regular octagon:

$$B^{(4)} = \mathrm{conv}\left\{\pm(1, 0), \pm(0, 1), \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\},$$

which induces the following octolinear norm:

$$||(x, y)||_{B^{(4)}} = (\sqrt{2} - 1) \cdot ||(x, y)||_{B(1)} + (2 - \sqrt{2}) \cdot ||(x, y)||_{B(\infty)}.$$

[19] Shang et al. [327] give a discussion for the specific case when the coordinate system is rotated without increasing the number of orientation directions.

[20] So this distance is sometimes called "Manhattan norm" or "Taxi-cab Geometry", see [160], [161].

[21] For a tutorial see [382].

With $B^{(\infty)}$ we describe the Euclidean unit ball.
For a survey about Steiner's Problem in λ-geometries see [45].[22]

II. For small values of λ we can determine the Steiner ratio analytically:

For $\lambda = 2$ it holds that $B^{(2)} = B(1)$, which we discussed above by saying $m_2(B^{(2)}) = \frac{2}{3}$. This fact is independent of whether 4.6.1 is true or not.

Consider $\lambda = 3$. Here, we first have

Lemma 4.8.1 *(Laugwitz [241]) Suppose that B is a unit ball in the plane. There is an affinely regular hexagon inscribed in B with vertices on the boundary of B.*

Proof. The first vertex p_1 may be arbitrarily chosen on bd$B$. We consider the function $\phi : \mathrm{bd}B \to I\!R$ defined by

$$\phi(v) = ||p_1 - v||_B.$$

Then $\phi(p_1) = 0$ and $\phi(-p_1) = 2$. Since ϕ is a continuous function and bd$B$ is a compact set, there is a point p_2 with $\phi(p_2) = 1$. Now it is easy to see that the points $p_1, p_2, p_2 - p_1, -p_1, -p_2$ and $p_1 - p_2$ are the vertices of the desired hexagon.

□

This gives immediately, (see below, the proof of 4.11.1) $m_2(C) \leq \frac{3}{4}$ for an affinely regular hexagon C. Since $B^{(3)}$ is such a hexagon, we obtain

Theorem 4.8.2 *Assume that 4.6.1 is true. Let B be an affinely regular hexagon in the plane. Then*

$$m_2(B) = \frac{3}{4}. \tag{4.67}$$

Altogether,

norm	$\lambda =$	Steiner ratio	source
rectilinear	2	$\frac{2}{3}$	[199]
hexagonal	3	$\frac{3}{4}$	4.8.2
octolinear	4	$\frac{2+\sqrt{2}}{4}$	conjecture in [246]
Euclidean	∞	$\frac{\sqrt{3}}{2}$	conjecture 4.6.1

III. By simple calculation

$$\cos\frac{\pi}{2\lambda} B(2) \subseteq B^{(\lambda)} \subseteq B(2). \tag{4.68}$$

With this in mind, good upper bounds for the Steiner ratio are given by

[22] In particular Swanepoel [344] shows the important fact that in λ-geometry degree four steiner points exist if and only if $\lambda \in \{2, 3, 4, 6\}$.

Theorem 4.8.3 *(Sarrafzadeh, Wong [319]) Assume that 4.6.1 is true. For the Steiner ratio of the planes with λ-geometry it holds that*

$$m_2(B^{(\lambda)}) \geq \frac{\sqrt{3}}{2} \cos \frac{\pi}{2\lambda}.$$

Proof. Let N be a finite set in A_2. Then,

$$\begin{array}{rcll} L(B^{(\lambda)})(\text{SMT for } N) & \geq & L(B^{(\infty)})(\text{SMT for } N) & \\ & = & L(B(2))(\text{SMT for } N) & \text{using (4.51)} \\ & \geq & \frac{\sqrt{3}}{2} \cdot L(B(2))(\text{MST for } N) & \text{with 4.6.1} \\ & = & \frac{\sqrt{3}}{2} \cdot L(B^{(\infty)})(\text{MST for } N) & \\ & \geq & \frac{\sqrt{3}}{2} \cos \frac{\pi}{2\lambda} \cdot L(B^{(\lambda)})(\text{MST for } N) & \end{array}$$

□

In view of 4.4.1, we also find[23],

Corollary 4.8.4 *Assume that 4.6.1 is true. For the Steiner ratio of the planes with λ-geometry*

$$m_2(B^{(\lambda)}) \leq \frac{\sqrt{3}}{2} \cdot \frac{1}{\cos \frac{\pi}{2\lambda}}. \tag{4.69}$$

IV. Now we have $m_2(B^{(2)}) = \frac{2}{3}$ and $m_2(B^{(3)}) = \frac{3}{4}$. But unfortunately we cannot extend the sequence so simply. In particular, it is an interesting question to investigate the equality in 4.8.3.

Theorem 4.8.5 *(Lee, Shen [247]) Assume that 4.6.1 is true. For the Steiner ratio of the planes with λ-geometry it holds that*

$$m_2(B^{(\lambda)}) = \frac{\sqrt{3}}{2} \cos \frac{\pi}{2\lambda},$$

if $\lambda \equiv 3 \bmod 6$, and

$$m_2(B^{(\lambda)}) = \frac{\sqrt{3}}{2},$$

if $\lambda \equiv 0 \bmod 6$, $\lambda \geq 6$.

Here, we find two phenomena:

- There are infinitely many different Banach-Minkowski planes which have the same Steiner ratio as the Euclidean plane.

[23] But note 4.9.5

- The Steiner ratio of the planes with λ-geometry is not a monotonically increasing function of the parameter λ.

Furthermore, when λ is not a multiple of 3, Lee, Shen [247] have shown that the lowers for $m_2(B^{(\lambda)})$ in 4.8.3 can be improved. They gives a discussion investigating equilateral triangles in λ-geometry, and find for several values of λ (assuming that 4.6.1 is true) numerically:

λ	degree	Steiner ratio
2	90	0.666...
3	60	0.75
4	45	0.853...
5	36	0.845...
6	30	0.866...
9	20	0.852...
10	18	0.863...
12	15	0.866...
15	12	0.861...
20	9	0.865...
30	6	0.866...
45	4	0.865...
60	3	0.866...
90	2	0.866...
180	1	0.866...
∞	Euclidean	0.866...

V. Until now, we considered affinely regular hexagons H, given by two points v and v' in the plane such that the origin o of the plane is not in the line aff$\{v, v'\}$. Then

$$H = \mathrm{conv}\{v, v', -v, -v', v - v', v' - v\}. \tag{4.70}$$

Now we will investigate hexagons in general. Let v_1, v_2 and v_3 be three pairwise linearly independent vectors satisfying $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = o$, for positive real numbers $\alpha_1, \alpha_2, \alpha_3$. It is not hard to see that

Lemma 4.8.6 *(Du et al. [131]) The hexagon is convex if and only if $\alpha_1, \alpha_2, \alpha_3$ satisfy the triangle inequality, i.e.*

$$\alpha_1 + \alpha_2 \geq \alpha_3, \alpha_1 + \alpha_3 \geq \alpha_2 \text{ and } \alpha_2 + \alpha_3 \geq \alpha_1. \tag{4.71}$$

The importance to consider hexagons comes from the following fact.

Theorem 4.8.7 *(Du et al. [131]) For Banach-Minkowski planes, normed by unit balls and hexagons, respectively, the following holds true:*

$$\inf\{m_2(B) : B \text{ a unit ball in } A_2\} = \inf\{m_2(C) : C \text{ a hexagon in } A_2\}. \tag{4.72}$$

The *proof* is included in the proof of 4.9.3. More exactly: Suppose that bd$B$ is smooth and strictly convex. Then every full SMT T consists of three sets of parallel segments. Now we show: For all planes $m_2(B) \geq m_0$ holds if and only if $m_2(C) \geq m_0$ holds for all planes normed by a hexagon.
The direction from left to right is clear. The other direction uses a smooth and strictly convex unit ball. Let v_1, v_2, v_3 be three vectors of unit length parallel to the edges of T, each starting in the origin. Let p_1, p_2, p_3 be the endpoints. Then $p_1, p_2, p_3 \in \mathrm{bd}B$ and

$$H = \mathrm{conv}\{p_1, -p_3, p_2, -p_1, p_3, -p_2\}$$

forms a hexagon inscribed in bd$B$. This gives the assertion.

□

4.9 Banach-Minkowski Planes

Consider the plane A_2 normed by the unit ball $B(1)$. Let $N = \{\pm(1,0), \pm(0,1)\}$, the nodes of the unit ball. It is easy to see that $\mu(N)(B(1)) = \frac{2}{3}$; and moreover, equality holds:

Theorem 4.9.1 *(Hwang [199], Salowe [315]) For the plane with rectilinear norm it holds*

$$m_2(B(1)) = \frac{2}{3} = 0.6666\ldots. \tag{4.73}$$

In view of the fact that all parallelograms are affine images of $B(1)$, we have

Corollary 4.9.2

$$m_2(B) = \frac{2}{3} = 0.6666\ldots, \tag{4.74}$$

whenever the unit ball B is a parallelogram.

Now, we are interested in the best lower bound for the Steiner ratio of any Banach-Minkowski plane. This bound must be at most 2/3. Moreover,

Theorem 4.9.3 *(Gao, Du, Graham [158]) For the Steiner ratio of Banach-Minkowski planes the following is true:*

$$m_2(B) \geq \frac{2}{3}.$$

Equality holds if B is a parallelogram.[24]

Idea of the *proof.* We use the Hausdorff distance between two unit balls:

$$\mathrm{dist}(B, B') = \inf\{r \geq 0 : B \subseteq B' + rB(2), B' \subseteq B + rB(2)\}. \tag{4.75}$$

[24] and only if?

This function forms a metric for the set of all unit balls[25].
Hadwiger [180] has shown

$$B + rB(2) \subseteq (1 + rh)B, \tag{4.76}$$

where

$$h = h(B) = \sup_{v \neq o} \frac{||v||_{B(2)}}{||v||_{DB}},$$

or in geometric terms, let $tB(2)$ be the largest among all Euclidean balls in B; then $h(B) = 1/t$, compare [83].
We find a strictly convex and smooth unit ball B' in any predetermined Hausdorff distance ϵ to B.

$$\frac{1}{1 + \epsilon h(B)} \cdot B' \subseteq B \subseteq (1 + \epsilon h(B')) \cdot B'.$$

Implying, with 4.4.1 in mind,

$$(1 + \epsilon h(B))(1 + \epsilon h(B')) \cdot m_2(B') \geq m_2(B) \geq \frac{1}{(1 + \epsilon h(B))(1 + \epsilon h(B'))} \cdot m_2(B').$$

$h(B')$ is a bounded number, and consequently, the difference between the Steiner ratios $m_2(B')$ and $m_2(B)$ is arbitrarily small.
Hence, we may assume that B is a strictly convex and smooth unit ball. Then we may use that

- Any Steiner point in an SMT has degree three. Moreover, without loss of generality, we may assume that the SMT is a full tree.
- Each full SMT consists of three sets of parallel segments.
- Any direction determines two unique directions such that three lines respectively in these directions intersect at vertices of an equilateral triangle, the so-called consistent triple.

Several extensive calculations with consistent triples yield the assertion.

□

With the proof of 4.9.3 one obtains a more general assertion.

Theorem 4.9.4 *(Gao, Du, Graham [158]) If there is a natural number n such that the bound $2/3$ is achieved by a set of n points, then $n = 4$, and B is a parallelogram.*

In contrast, an upper bound is given by the following theorem. The *proof* starts the same as 4.9.3. After many combinatorial restrictions on the structure of an SMT, the result follows by constructing a set N of given points achieving that bound.

[25] And is sometimes called Blaschke's Nachbarschaftsmaß.

Theorem 4.9.5 *(Du et al. [131]) For any unit ball B in the plane*

$$m_2(B) \leq \frac{\sqrt{13}-1}{3} = 0.8685\ldots. \tag{4.77}$$

There is no unit ball known which makes the inequality in 4.9.5 to an equality. Does the Euclidean metric indeed have the greatest Steiner ratio of all Banach-Minkowski metrics?

Conjecture 4.9.6 *For any unit ball B in the plane the following is true:*

$$m_2(B) \leq \frac{\sqrt{3}}{2} = 0.8665\ldots. \tag{4.78}$$

4.10 The Steiner ratio of $\mathcal{L}_p^3$

In this section we will determine upper bounds for the Steiner ratio of three-dimensional spaces with p-norm: $m(3,p) = m_3(B(p)) = m(\mathcal{L}_p^3)$.
Considering the four points

$$\begin{array}{rcl} v_1 &=& (1,0,0), \\ v_2 &=& (0,1,0), \\ v_3 &=& (0,0,1) \quad \text{and} \\ v_4 &=& (1,1,1) \end{array}$$

which build an equilateral set in the three-dimensional space with mutual distances $\sqrt[p]{2} = \rho(v_i, v_j)$, for $i \neq j$, we find

Theorem 4.10.1 *(Albrecht [3], Albrecht, Cieslik [8], Cieslik [92]) Let $1 < p < \infty$ and let q be the conjugate of p. Then we have for the Steiner ratio of $\mathcal{L}_p^3$*

$$m(3,p) \leq \left\{ \begin{array}{ccl} \frac{1}{3}\left(2^{-1/p} + (2^q - 1)^{1/q}\right) & : & 1 < p \leq \frac{\log 3}{\log 3 - \log 2} = 2.70931\ldots \\ \left(\frac{2}{3}\right)^{1/q} & : & \textit{otherwise} \end{array} \right.$$

On the other hand, using six points

$$\begin{array}{rcl} v_1 &=& (x, x-1, 1-x), \\ v_2 &=& (x, x, 2-x), \\ v_3 &=& (1,0,1), \\ v_4 &=& (0,0,0), \\ v_5 &=& (0,1,1) \quad \text{and} \\ v_6 &=& (x-1, x, 1-x), \end{array}$$

and adding four Steiner points, we have

Theorem 4.10.2 *(Albrecht [3], Albrecht, Cieslik [8], Cieslik [92]) Let p and q be reals with $1 < p < \infty$, $1/p + 1/q = 1$; and let x_0 be the unique determined zero of the function f with*

$$f(x) = x^p + 2(x-1)^p - 2$$

in the range $(1, 2)$. Then the Steiner ratio of $\mathcal{L}_p^3$ can be estimated by

$$m(3,p) \leq \begin{cases} \frac{1}{5}\left((2^q-1)^{1/q} + \left(\frac{1}{2}\right)^{1/p} + \left(\frac{3}{2}\right)^{1/p} x_0\right) & : \quad 1 < p \leq \frac{\log 3}{\log 3 - \log 2} \\ \frac{1}{5}\left(\frac{3}{2}\right)^{1/p}(x_0+2) & : \quad \frac{\log 3}{\log 3 - \log 2} < p < \infty \end{cases}$$

Using 4.10.2 for $p = \infty$ gives the value $3/5 = 0.6$ for the Steiner ratio, but here we have with help of another consideration, namely 4.14.1, the better bound $m(3,\infty) \leq 4/7 = 0.5714\ldots$.

Theorem 4.10.3 *(Albrecht, Cieslik [7]) If the conjectures 4.3.2 and 4.12.3 are true, then*

$$m(3,p) \geq 0.5212\ldots. \tag{4.79}$$

The *proof* combines 4.4.2 and 4.5.3 for $d = 3$ and uses several simple calculations.

□

4.11 The Range of the Steiner Ratio

An interesting problem, but which seems very difficult, is to determine the range of the Steiner ratio for d-dimensional Banach-Minkowski spaces, depending on the value d. More exactly, determine the best possible reals c_d and C_d such that

$$c_d \leq m_d(B) \leq C_d, \tag{4.80}$$

for all unit balls B of A_d, $d = 1, 2, 3, \ldots$. That means:

$$\begin{aligned} c_d &= \inf\{m_d(B) : B \text{ a unit ball in } A_d\} \quad \text{and} \\ C_d &= \sup\{m_d(B) : B \text{ a unit ball in } A_d\}. \end{aligned}$$

Both, the numbers C_d and c_d, are attained by certain Banach-Minkowski spaces. This follows from the continuity of the Steiner ratio as a function of the space and the Blaschke selection theorem.

Of course, $C_1 = 1$, but C_2 is essentially less, since

Theorem 4.11.1 *In any Banach-Minkowski space $M_d(B)$ where $d \geq 2$, there is a three point set N such that the SMT for N is strictly shorter than an MST for N.*

For a *proof* we start with the observation 4.8.1 that it is possible to inscribe a "regular" hexagon into the unit ball of any Banach-Minkowski plane. Here, "regular" has two meanings: a) The hexagon is regular in the sense that all edges have the same length; and b) It is also affinely regular - an affine image of an Euclidean regular hexagon.
Let $M_2(B)$ be a Banach-Minkowski plane. In view of 4.8.1 let C be an inscribed affinely regular hexagon for the unit ball B such that the nodes $p_1, ..., p_6$ of C are placed in this order on the boundary of B. Now we distinguish two cases.

1. $B = C$.
Up to isometry, we may assume that

$$B = \mathrm{conv}\{(1,1), (-1,-1), (1,0), (-1,0), (0,1), (0,-1)\}, \tag{4.81}$$

which implies that

$$||(x_1, x_2)||_B = \max\{|x_1|, |x_2|, |x_1 - x_2|\}. \tag{4.82}$$

It is easy to see that the set $N = \{p_1, p_3, p_5\}$ has an MST of length 4 and an SMT of length at most 3.

2. Suppose that C is a proper subset of B.
Then there is a point p in $\mathrm{bd}B \setminus C$. Without loss of generality we may assume that p lies in the cone spanned by p_1, o, p_2. Let q be the only element of the intersection $\underline{p_1 p_2}$ and $\underline{op}$. Then $||q||_B < 1$. Consequently, an SMT for $\{o, p_1, p_2\}$ is strictly shorter than an MST.

This completes the proof for $d = 2$.
For the higher-dimensional case we use 4.3.7.

□

Theorem 4.11.2 *A Banach-Minkowski space $M_d(B)$ has Steiner ratio 1 if and only if $d = 1$.*

What can we say about higher dimensions? At first view it seems that it will be simpler to show the upper rather than the lower bound. In fact, this is not the case.

$0.612\ldots$	$\le c_2 \le C_2 \le$	$0.9036\ldots$	Cieslik, 1990, [73]
$0.623\ldots$	$\le c_2 \le C_2 \le$	$0.8686\ldots$	Du, Gao, Graham, Liu, Wan, 1993, [131]
$0.666\ldots$	$\le c_2$		Gao, Du, Graham, 1995, [158]

Conjecture 4.11.3 $C_d = m(d, 2)$, *where $m(d, 2)$ denotes the Steiner ratio of the d-dimensional Euclidean space.*

This conjecture is open for all values of d, also in the planar case, for which we only know $m(2, 2) \le C_2 \le \frac{\sqrt{13}-1}{3}$, see 4.9.5, compare [125], [129] and [131].
On the other hand, $c_1 = 1$ and c_2 is essentially less than one, but

Conjecture 4.11.4 $c_d > 1/2$.

That means that there is no Banach-Minkowski space in which the Steiner ratio achieves the smallest possible value 1/2. But note that for each positive real number ϵ there is a space $M_d(B)$ such that $m_d(B) \leq 1/2 + \epsilon$.
The conjecture is open, except for the planar case, for which we know $c_2 = \frac{2}{3}$, see 4.9.3, compare [158].

As an example, consider $\mathcal{L}_p^3$. $m(3,p)$ cannot be greater than $m(2,p)$, since $\mathcal{L}_p^2$ is a subspace of $\mathcal{L}_p^3$. Hence, $m(3,p) \leq 0.86602\ldots$. If 4.11.3 is true, then $m(3,p) \leq 0.78419\ldots$. On the other hand, in view of 4.10.3, it holds $m(3,p) > 1/2$.

4.12 The Steiner Ratio of Euclidean Spaces

I. In the d-dimensional Euclidean space, we consider the set N of $d+1$ nodes of a regular simplex with exclusively edges of unit length. Then an MST for N has the length d. It is easy to compute that the sphere that circumscribes N has the radius

$$R(N) = \sqrt{d/(2d+2)}. \tag{4.83}$$

With the center of this sphere as Steiner point, we find a tree T interconnecting N with the length

$$L(B(2))(T) = (d+1)R(N). \tag{4.84}$$

Hence, we find by (4.83) and (4.84) the following nontrivial upper bound:

$$\mu(N) \leq \frac{(d+1)\sqrt{\frac{d}{2d+2}}}{d} = \sqrt{\frac{d+1}{2d}}. \tag{4.85}$$

Theorem 4.12.1 *The Steiner ratio of the d-dimensional Euclidean space can be bounded as follows:*

$$m(d,2) \leq \sqrt{\frac{1}{2} + \frac{1}{2d}}. \tag{4.86}$$

In the proof we used a Steiner point of degree $d+1$, but it is well-known, see 4.2.1, that all Steiner points in an SMT in Euclidean space of any dimension are of degree 3. Hence, we may assume that we can find better bounds than in 4.12.1; see 4.10.1.

II. A generalized conjecture, posed by Gilbert and Pollak, stated that the Steiner ratio of any Euclidean space is achieved when the given points are the nodes of a regular simplex. The regular simplex is a generalization, to the d-dimensional Euclidean space, of the two-dimensional triangle and the 3-dimensional tetrahedron. It has $d+1$ nodes and the mutual distances between the nodes of the simplex are equal. In 1992, Smith [329] showed that the generalized Gilbert-Pollak conjecture is false for the dimension d with $3 \leq d \leq 8$. Moreover, the conjecture is disproved in general by

Theorem 4.12.2 *(Chung, Gilbert [65], Smith [329] and Du, Smith [135]) The Steiner ratio of the d-dimensional Euclidean space is bounded as follows:*

dimension	*upper bound by Chung, Gilbert*	*upper bound by Smith*	*upper bound by Du, Smith*
$=2$	$0.86602\ldots$		
$=3$	$0.81305\ldots$	$0.81119\ldots$	$0.78419\ldots$
$=4$	$0.78374\ldots$	$0.76871\ldots$	$0.74398\ldots$
$=5$	$0.76456\ldots$	$0.74574\ldots$	$0.72181\ldots$
$=6$	$0.75142\ldots$	$0.73199\ldots$	$0.70853\ldots$
$=7$	$0.74126\ldots$	$0.72247\ldots$	$0.70012\ldots$
$=8$	$0.73376\ldots$	$0.71550\ldots$	$0.69455\ldots$
$=9$	$0.72743\ldots$	$0.71112\ldots$	$0.69076\ldots$
$=10$	$0.72250\ldots$		$0.68812\ldots$
$=11$	$0.71811\ldots$		$0.68624\ldots$
$=20$	$0.69839\ldots$		
$=40$	$0.68499\ldots$		
$=80$	$0.67775\ldots$		
$=160$	$0.67392\ldots$		
$\to\infty$	$0.66984\ldots$		

The first column was computed by Chung and Gilbert considering regular simplices. Here, Du and Smith [135] showed that the regular d-simplex cannot achieve the Steiner ratio if $d > 2$. That means that these bounds cannot be the Steiner ratio of the space when $d > 2$.
The second column given by Smith investigates regular octahedra, respectively cross-polytopes. Note that it is not easy to compute an SMT for the nodes of an octahedra.
In the third column the ratio of sausages is used, whereby a sausage is constructed by

1. Start with a ball (of unit diameter) in $\mathcal{L}_2^d$;
2. Successively add balls such that the n'th ball you add is always touching the $\min\{d, n-1\}$ most recently added balls.

This procedure uniquely[26] defines an infinite sequence of interior-disjoint numbered balls. The centers of these balls form a discrete point set, which is called the (infinity) d-sausage $N(\infty, d)$. The first n points of the d-sausage will be called the "n-point d-sausage" $N(n, d)$. Note that $N(d+1, d)$ is a d-simplex if $d \geq 3$.
Du and Smith [135] present many properties of the d-sausage, in particular, that

$$u(d) := \mu(N(\infty, d)) = \frac{L(\text{SMT for } N(\infty, d))}{L(\text{MST for } N(\infty, d))} \tag{4.87}$$

[26] up to congruence

is a strictly decreasing function of the dimension d.[27] Hence, $u(d)$, $d = 2, 3, \dots$ is a convergent sequence, but the limit is still unknown.
It seems that there does not exist a finite set of points in the d-dimensional Euclidean space, $d \geq 3$, which achieves the Steiner ratio $m(d, 2)$. But, if such set in spite of it exists, then it must contain exponentially many points. More exactly: Smith, Smith [334] investigate sausages in the three-dimensional Euclidean space to determine the Steiner ratio and following, they conjectured that

Conjecture 4.12.3 *For the Steiner ratio of the three-dimensional Euclidean space*

$$\begin{aligned} m(3,2) &= \sqrt{\frac{283}{700} - \frac{3\sqrt{21}}{700} + \frac{9\sqrt{11-\sqrt{21}}\sqrt{2}}{140}} \\ &= 0.78419\dots. \end{aligned}$$

Even if the conjectured value turns out to be incorrect, it acts as a good upper bound on the true value of the Steiner ratio $m(3, 2)$.
Further going Smith [335] consider Steiner's Problem for an infinite (but countable) number of given points which form a tripel helix.
These investigations are helpful to discuss the following problem: One of the key issues in biochemistry today is predicting the three-dimensional structure of proteins from the primary sequence of amino acids. Steiner's Problem in the three-dimensional Euclidean space might help explain the reason for these long molecular chains. In order to examine this potential application area and others related to it, possible linkages between the objective function of Steiner's Problem and objective functions of these applications in biochemical sciences need to be examined, see [273], [274], [275], [334], and [353].

III. Moreover, Du and Smith used the theory of packings to get the following result.

Theorem 4.12.4 *(Du, Smith [135]) Let N be a finite set of n points in the d-dimensional Euclidean space $M_d(B(2))$, $d \geq 3$, which achieves the Steiner ratio $m_d(B(2))$ of the space. Then*

$$n \geq \left\lceil \frac{1}{2} \cdot \sqrt{f\left(\frac{\pi}{3}, d\right)} \right\rceil + 1,$$

where

$$f(\theta, d) = \frac{2I_{d-2}(\pi/2)}{I_{d-2}(\theta)}$$

[27] Here, we use a generalization of Steiner's Problem to sets of infinitely many points. This is simple to understand. For a finite number of points it is shown that

$$\mu(N(2d+1,d)) \leq \mu(N(d+1,d)),$$

which is a finite version of

$$\mu(N(\infty,d)) \leq \mu(N(d+1,d)),$$

for $d > 1$.

and

$$I_m(x) = \int_0^x (\sin u)^m \, du.$$

4.12.4 implies that the number n grows at least exponentially in the dimension d. Some numbers are computed:

$d =$	n is at least
50	53
100	2218
200	3481911
500	10^{16}
1000	$5 \cdot 10^{31}$

IV. A lower bound for the Steiner ratio of Euclidean spaces is given by

Theorem 4.12.5 *(Graham, Hwang [171]) For the Steiner ratio of any Euclidean space it holds*

$$m(d,2) \geq \frac{1}{\sqrt{3}} = 0,57735\ldots.$$

Proof. Let N be a set of n points in $M_d(B(2))$.
All Steiner points of an SMT are of degree three implies that it is sufficient only to consider SMT's $T = (V, E)$ which are full trees for N.
In view of 2.5.5, there is a Steiner point q in T with two neighbors v, v' in N. Without loss of generality, we may assume that $||v - q|| \geq ||v' - q||$. Using the cosine law, it is easily verified that $||v - q||/||v - v'|| \geq 1/\sqrt{3}$.
Let T' be an SMT and T'' an MST for the set $N \setminus \{v\}$. Then by an induction on n

$$\begin{aligned}
\frac{L(T)}{L(\text{MST for } N)} &\geq \frac{||v-q|| + L(T \text{ without the edge } \underline{vq})}{||v-v'|| + L(T'')} \\
&\geq \frac{||v-q|| + L(T')}{||v-v'|| + L(T'')} \\
&\geq \min\left\{\frac{||v-q||}{||v-v'||}, \frac{L(T')}{L(T'')}\right\} \geq \frac{1}{\sqrt{3}}.
\end{aligned}$$

□

This lower bound is improved by a lot of geometric investigations.

Theorem 4.12.6 *(Du [127])* $m(d,2) \geq 0,615\ldots.$

V. $\{m(d,2)\}_{d=1,2,\ldots}$ is a monotone decreasing, bounded, and consequently, convergent sequence. By 4.12.1 we have the bound $1/\sqrt{2} = 0.70710\ldots$ for the limit when the dimension d runs to infinity, which can be improved by the consideration of a sequence of trees on regular simplices whose lengths go decreasingly.

Theorem 4.12.7 *(Chung, Gilbert [65])*

$$\lim_{d\to\infty} m(d,2) \leq \frac{\sqrt{3}}{4-\sqrt{2}} = 0.66984\ldots.$$

Sketch of the *proof.* We consider a set of points created by the nodes of a regular simplex. Such a simplex is a generalization, to the d-dimensional Euclidean space, of the 2-dimensional triangle and a 3-dimensional tetrahedron. It has $n = d+1$ nodes $N = \{v_1, \ldots, v_n\}$. We call the simplex regular if all the distances $||v_i - v_j||$ for $i \neq j$ are equal.
Take $v_1 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)$, such that each v_i has a "1" in the ith coordinate and all other coordinates are "0". Then $||v_i - v_j|| = \sqrt{2}$. Since all trees for $v_1, \ldots, v_n$ are of equal length we obtain

$$L(\text{MST for}) = \sqrt{2} \cdot d. \tag{4.88}$$

Remember that a Steiner tree with $n-2$ Steiner points is called a full tree, and each Steiner tree can be decomposed by full trees. Therefore,

$$n = 2^{r_1} + 2^{r_2} + \ldots \tag{4.89}$$

such that N is partitioned into subsets containing $2^{r_1}, 2^{r_2}, \ldots$ vertices.
Each of the 2^{r_k} vertices will be connected through a binary tree including several Steiner points. In particular, a core Steiner point $\frac{1}{n} \cdot (1, \ldots, 1)$ will be used, which is the centroid of the entire simplex.
After several calculations we find the desired bound.

□

VI. The so-called Einstein-Riemann metric, which is used in differential geometry and in the theory of relativity, is defined with a positive definite matrix $\Psi = (p_{ij})_{i,j=1,\ldots,d}$ by

$$||v||_\Psi = (\Psi v, v)^{1/2} = \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{d} p_{ij} x_i x_j}, \tag{4.90}$$

where $v = (x_1, \ldots, x_d)$. For $\Psi = \mathcal{I}$ the norm $||.||_\Psi$ is the Euclidean one.

Horn, Johnson [194] shows that for a positive definite matrix Ψ and an integer $k \geq 1$ there exists a unique positive Hermitian matrix Φ such that $\Phi^k = \Psi$. Moreover, rank Φ = rank Ψ. In other terms, each positive definite matrix has a unique k'th root for all $k = 1, 2, \ldots$. Hence, for $k = 2$:

Lemma 4.12.8 *(Horn, Johnson [194]) Let Ψ be a positive definite matrix. Then there exists a unique nonsingular matrix Φ such that $\Psi = \Phi^\star \Phi$.*

With 4.12.8 in mind, we find

$$||v||_\Psi^2 = (\Psi v, v) = (\Phi^\star \Phi v, v) = (\Phi v, \Phi v) = ||\Phi v||_{\mathcal{I}}^2.$$

This implies

$$||v||_\Psi = ||\Phi v||_{B(2)}, \tag{4.91}$$

which says, compare (4.15), that Φ is an isometry to the Euclidean space. In view of 4.1.2 and 4.3.4, we have that the Steiner ratio of a d-dimensional Einstein-Riemann space depends only on the dimension d, and not on the specific choice of the matrix.

Theorem 4.12.9 *Let $M(d,\Psi)$ be a d-dimensional Einstein-Riemann space normed by the positive definite matrix Ψ. Then*

$$m(M(d,\Psi)) = m(d,2),$$

where $m(d,2)$ denotes the Steiner ratio of the d-dimensional Euclidean space.

4.13 The Steiner Ratio of $\mathcal{L}_p^d$

We will determine upper bounds for the Steiner ratio of d-dimensional $\mathcal{L}_p$-spaces, abbreviated by $m(d,p)$, that is $m(d,p) = m(\mathcal{L}_p^d)$.

I. Let $\Delta_{i,j}$ be the Kronecker-symbol. Then a d-dimensional cross-polytope is the convex hull of

$$\begin{aligned} N &= \{v_i = (x_{i,1},\ldots,x_{i,d}) \ : x_{i,j} = \Delta_{i,j}, i,j = 1,\ldots,d\} \\ &\quad \cup\{v_i = -v_{i-d} : i = d+1,\ldots,2d\} \end{aligned}$$

which contains $2d$ points. For $1 \le i < j \le 2d$ we have

$$\rho(v_i,v_j) = \left\{ \begin{array}{rcl} 2 & : & j = i+d \\ 2^{1/p} \le 2 & : & \text{otherwise} \end{array} \right.$$

and consequently

$$L(\text{MST for } N) = (2d-1)\cdot 2^{1/p}.$$

If we add the orign o, we find a shorter tree. More exactly,

$$L(\text{SMT for } N) \le L(\text{MST for } N\cup\{o\}) = 2d,$$

using $\rho(v_i,o) = 1$ for $i = 1,\ldots,2d$. Hence, it was proved

Theorem 4.13.1 *(Albrecht [3], [92]) For the Steiner ratio of the space $\mathcal{L}_p^d$ it holds*

$$m(d,p) \le \frac{2d}{2d-1}\cdot\left(\frac{1}{2}\right)^{1/p}.$$

Obviously, the bound given in 4.13.1 is monotonically increasing in the value p. Hence, we may assume that for "big" p we will find a better bound using the dual polytope of a cross-polytope.[28] And indeed,

[28] With 4.1.8 in mind.

Theorem 4.13.2 *(Albrecht [3], [92]) For the Steiner ratio of the space $\mathcal{L}_p^d$ it holds*

$$m(d,p) \le \frac{2^{d-1}}{2^d - 1} \cdot d^{1/p}.$$

Proof. Let N be the set of the 2^d points $(\pm 1, \dots, \pm 1)$. Then conv$N$ is a d-dimensional hypercube. The mutual distances between two different points in N are at least 2. It is not hard to see that an MST has length $2 \cdot (2^d - 1)$.
Let $T = (N \cup \{o\}, \{\underline{ov} \ : v \in N\})$, using $\rho(o,v) = d^{1/p}$ for any $v \in N$. Then

$$m(d,p) \le \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} \le \frac{L(T)}{2(2^d-1)} = \frac{2^d \cdot d^{1/p}}{2(2^d-1)}.$$

□

II. To find general bounds for $m(d,p)$ we follow an idea by Liu, Du [252]. Recall that the p-norm satisfy monotonicity properties: 4.1.7. Consequently,

$$||v||_{B(q)} \le ||v||_{B(r)} \le d^{1/r-1/q} ||v||_{B(q)}, \tag{4.92}$$

if $1 \le r \le q \le \infty$.[29] Application of (4.92) in the sense of 4.4.1 gives

$$\begin{array}{rcl} m(d,p) & \ge & d^{1/q-1/p} m(d,q) \quad \text{and} \\ m(d,p) & \ge & d^{1/p-1/r} m(d,r). \end{array}$$

Multiplying both together, we obtain

$$m(d,p)^2 \ge d^{1/q-1/r} \cdot m(d,r) \cdot m(d,q). \tag{4.93}$$

Similar the converse inequality.

Theorem 4.13.3 *Suppose $1 \le r \le p \le q \le \infty$. Then*

$$d^{1/q-1/r} \cdot m(d,r) \cdot m(d,q) \le m(d,p)^2 \le d^{1/r-1/q} \cdot m(d,r) \cdot m(d,q). \tag{4.94}$$

For $d = 3$ using 4.13.3 with $r = p, q = 2$ and $r = 2, p = q$ we obtain again 4.10.3.

4.14 Cubes

I. Other than the bound given in 4.13.1, the bound in 4.13.2 is monotonically decreasing in the value p. Hence, if p runs to infinity, we have

Theorem 4.14.1

$$m(d,\infty) \le \frac{2^{d-1}}{2^d - 1}.$$

[29] This equation can be written in terms of the Banach-Mazur distance, see 4.5.3.

Hence, the Steiner ratio of $M_d(B(\infty))$ tends to 1/2 if the dimension d runs to infinity.
Comparing 4.3.1 and 4.14.1 the Steiner ratio of $\mathcal{L}_1^d$ runs with $O(\frac{1}{d})$ and the Steiner ratio of $\mathcal{L}_\infty^d$ with $O(\frac{1}{2^{d+1}})$ to $\frac{1}{2}$. Numerically:

dimension d	$m(d,1) \leq$	$m(d,\infty) \leq$
2	$0.66666\ldots$	$0.66666\ldots$
3	0.6	$0.57142\ldots$
4	$0.57142\ldots$	$0.53333\ldots$
5	$0.55555\ldots$	$0.51612\ldots$
6	$0.54545\ldots$	$0.50793\ldots$
$\vdots$	$\vdots$	$\vdots$
10	$0.52631\ldots$	$0.50048\ldots$
$\vdots$	$\vdots$	$\vdots$
$\to\infty$	0.5	0.5,

which says that $m(d,\infty)$ runs faster to 1/2 than $m(d,1)$. This is not a surprise, since there is an isometric embedding of $\mathcal{L}_1^d$ into $\mathcal{L}_\infty^{2^d}$, see 4.16.1(a). And in view of 6.2.2 we have $m(2^d,\infty) \leq m(d,1) \leq d/(2d-1)$.

Conjecture 4.14.2 $c_d = m(d,\infty)$.

Note that the conjectures 4.11.4 and 4.14.2 are independent of each others, unless there does not exist a Banach-Minkowski space with Steiner ratio 0.5.

II. The unit ball $B(\infty)$ is the hypercube $[-1,1]^d$ with 2^d nodes. As supplement consider the unit ball H defined by

$$H = \mathrm{conv}([0,1]^d \cup [-1,0]^d) \tag{4.95}$$

and inducing the norm

$$||(x_1,\ldots,x_d)|| = \max\{x_i : x_i \geq 0\} - \min\{x_i : x_i \leq 0\}. \tag{4.96}$$

H is a convex polytope with exactly $2(2^d-1)$ nodes. Consider "half" of these, namely $v_1,\ldots,v_{2^d-1}$, which mutually differ in exact two coordinates. Then $||v_i - v_j|| = 2$, for all $i,j = 1,\ldots,2^d-1$, $i \neq j$. Hence,

$$L(\mathrm{MST}) = 2 \cdot (2^d - 1 - 1) = 2^{d+1} - 4.$$

On the other hand, with Steiner point o we find a tree of length $2^d - 1$.

Theorem 4.14.3 *Let H be defined as above. Then*

$$m_d(H) \leq \frac{2^d - 1}{2^{d+1} - 4}. \tag{4.97}$$

This bound for $m_d(H)$ is a little bit greater than 4.14.1, namely

dimension d	H is an	$m_d(H) \leq$
2	affinely regular hexagon	3/4
3	rhombic dodecahedron	7/12
4		15/28
$\vdots$	$\vdots$	$\vdots$
$\to\infty$		0.5

4.15 Equilateral Sets

Saying that the Steiner ratio is a measure of the geometry of the space related to its combinatorial properties forces the interest of other measures.[30]
We investigate quantities which are in relation to the distances in Banach-Minkowski spaces. Particularly, we are interested in the diameter of bounded sets and, moreover, in pairs of points in such sets which achieve this value.

I. For a bounded set X in a Banach-Minkowski space $M_d(B)$, we define the diameter as

$$D_B(X) = \sup\{||v - v'||_B : v, v' \in X\} \tag{4.98}$$

and the (circum-) radius as

$$R_B(X) = \inf\{r \geq 0 : v_o \in A_d, v_0 + rB \supseteq X\}. \tag{4.99}$$

Remark 4.15.1 *(Jung [217], [249]) Let X be a bounded set in the d-dimensional Euclidean space $M_d(B(2))$. Then*

$$R_{B(2)}(X) \leq \sqrt{\frac{d}{2(d+1)}} \cdot D_{B(2)}(X). \tag{4.100}$$

The quantity

$$J_d(B) = \sup\left\{\frac{R_B(X)}{D_B(X)} : X \text{ is a bounded set in } M_d(B)\right\} \tag{4.101}$$

is a geometrical constant, called the Jung number (of $M_d(B)$). It holds, see [249]: $1/2 \leq J_d(B) \leq d/(d+1)$.

Theorem 4.15.2 *(Cieslik [83]) There are the following interrelations between the Jung number and the Steiner ratio of Banach-Minkowski spaces $M_d(B)$:*

a) $m_2(B) \leq \frac{3}{2} \cdot J_2(B)$.

[30] Here, we will use only one of such relations. Several other geometric considerations imply estimates for the Steiner ratio, compare [92] and [210]. We may expect that further investigations about the combinatorial geometry of Banach-Minkowski spaces will give new results.

b) *If there is a regular simplex with unit edge length in* $M_d(B)$*, then*

$$m_d(B) \leq \left(1 + \frac{1}{d}\right) \cdot J_d(B).$$

II. Of course, there is an equidistant set of $d+1$ points in the Euclidean space $M_d(B(2))$, namely the nodes of a regular simplex.[31] Does every d-dimensional Banach-Minkowski space admit an equilateral simplex? At first glance, it may appear, that the answer is "yes", but it is an open question, even if the unit ball is smooth and if $d = 4$. Petty [292] shows that any set of equidistant points in a d-dimensional Banach-Minkowski space has at most the cardinality 2^d, and equality is attained only when the unit ball is affinely equivalent to the d-dimensional hypercube. Also, for sufficiently large dimension d in any d-dimensional affine space there exists a strictly convex unit ball B such that there is an equidistant set in the space $M_d(B)$ with at least $(1.02)^d$ points. For all these facts compare [156], [243] or [339].

III. We will use the idea of the existence of a regular simplex similar to in Euclidean spaces. For our investigations we have the following facts: Let $1 < p < \infty$ and $d \geq 3$. Then there are at least $d+1$ equidistant points in the space $\mathcal{L}_p^d$. This can be seen with the following considerations: Consider d points with exactly one coordinate equal to 1 and all the others equal to 0; that is, for $i = 1, \ldots, d$ let $v_i = (x_{i,1}, \ldots, x_{i,d})$ with

$$x_{i,j} = \begin{cases} 1 & : \quad i = j \\ 0 & : \quad \text{otherwise} \end{cases}$$

It is $||v_i - v_j|| = 2^{1/p}$ for all $1 \leq i < j \leq d$.
For the point $v = (x, \ldots, x)$ it holds that $||v - v_i|| = ||v - v_j||$ for all $1 \leq i, j \leq d$.
To create $||v - v_i|| = 2^{1/p}$ the value x has to fulfill the equation

$$((d-1)|x|^p + |1 - x|^p)^{1/p} = 2^{1/p}.$$

This we can realize by the fact that the function $f : [0,1] \to I\!R$ with

$$f(x) = ((d-1)x^p + (1-x)^p)^{1/p} - 2^{1/p}$$

has exactly one zero in $[0,1]$.

Theorem 4.15.3 *(Albrecht [3], Albrecht, Cieslik [4], [5]) Let* $1 < p < \infty$ *and* $d \geq 3$*. Then*

$$m(d,p) \leq \frac{d+1}{2d} \cdot \left(\frac{d}{2}\right)^{1/p}.$$

IV. Extending this method,

Theorem 4.15.4 *(Albrecht [3], Albrecht, Cieslik [4], [5]) Let* $1 < p < \infty$*. Then*

$$m(d,p) < \frac{d+1}{d} \cdot \left(\frac{1}{2}\right)^{1/p}.$$

[31] Remember that we used this fact in the proof of 4.12.1.

Proof. Let N be the set with the $d+1$ points constructed above, and let w be the "center" of this construction. Then $L(\text{MST for } N) = d$ and $L(\text{SMT for } N) \leq (d+1) \cdot 2^{-1/p}$.

□

This bound is not sharp, since the estimation of the distance of the points to the center is too inefficient, at least for small dimensions. On the other hand, we only use one additional point, and it is to be assumed that more than one of such points will decrease the length.

Now, we compare the bounds given in 4.15.3 and 4.15.4. Obviously,

$$\frac{d+1}{2d} \cdot \left(\frac{d}{2}\right)^{1/p} \leq \frac{d+1}{d} \cdot \left(\frac{1}{2}\right)^{1/p} \tag{4.102}$$

holds if and only if

$$d \leq 2^p. \tag{4.103}$$

Hence,

Observation 4.15.5 *Looking for the Steiner ratio of high dimensional $\mathcal{L}_p$-spaces, we only have to consider the bound given in 4.15.4, more exactly, when (4.103) is satisfied.*

More relation between the Steiner ratio of different $\mathcal{L}_p$-spaces we can find in [210].[32]

Corollary 4.15.6

$$\lim_{d\to\infty} m(d,p) \leq \left(\frac{1}{2}\right)^{1/p} \tag{4.104}$$

for any real number $1 \leq p \leq \infty$.

4.16 The Steiner Ratio of $\mathcal{L}_{2k}^d$

It is obvious that all one-dimensional Banach spaces are isometric to each other so that $M_1(B(p))$ can be embedded into $M_{d'}(B(q))$ for any dimension d' and for any real number $q \geq 1$. Also, it is clear that $M_d(B(p))$ can be embedded into $M_{d'}(B(p))$ for any $d' \geq d$ and any p. This, together with 4.3.7, implies that the function $m(d,p)$ is monotonically decreasing with respect to the dimension d:

$$1 = m(1,p) > m(2,p) \geq m(3,p) \geq m(4,p) \geq \ldots \geq \lim_{d\to\infty} m(d,p) \geq 0.5.$$

(In view of 4.1.5 and 4.3.5, we may assume that the inequalities are strict for $p \neq 1, 2, \infty$.)
We saw that the determination of the values $m(d,p)$ is a nontrivial question. Insofar it will be very difficult to describe the behavior of the function $\underline{m}(p) = \lim_{d\to\infty} m(d,p)$.

[32] A nontrivial question: For which values p is the inequality (4.103) senseless?

We will attack this question for specific spaces. For instance, $0.615\ldots \leq \underline{m}(2) \leq 0.669\ldots$. Banach [18] proved that if $p \neq 2$ then each isometric embedding from a space $M_d(B(p))$ into itself is a permutation of the basis vectors followed by a sign change of some of these vectors. On the other hand, consider the following example [229]:

$$(x^2+y^2)^2 = \frac{8}{9}\left(x^4 + \left(\frac{\sqrt{3}y-x}{2}\right)^4 + \left(\frac{\sqrt{3}y+x}{2}\right)^4\right)$$

yields an isometric embedding from $M_2(B(2))$ into $M_3(B(4))$. That means that the unit ball $B(4)$ in the three-dimensional affine space contains a circular section. Hence, $m(3,4) \leq m(2,2) \leq \sqrt{3}/2 = 0.86602\ldots$.
Clearly, we are interested in the cases that $d' > d \geq 2$ and $p \neq q$. Unfortunately, isometric embeddings are rare:

Remark 4.16.1 *For isometric embeddings between spaces with p-norm the following holds true:*

- **a)** *(Lyubich, Vaserstein [258]) An isometric embedding $M_d(B(\infty)) \to M_{d'}(B(q))$ exists if and only if $d = 2$ and $q = 1$. An isometric embedding $M_d(B(p)) \to M_{d'}(B(\infty))$ exists if and only if $p = 1$ and $d' \geq 2^{d-1}$.*
- **b)** *(Lyubich, Vaserstein [259]) If $p, q \neq \infty$ and there is an isometric embedding from $M_d(B(p))$ into $M_{d'}(B(q))$ then $p = 2$, and q is an even integer.*

In general, it is not simple to construct isometric embeddings, since 4.16.1(b) gives only a necessary condition.[33] Now, suppose that q is an even integer. It is convenient to define the Waring number $W(d,q)$ as follows:

$$\begin{aligned} W(d,q) = \min\{d' \in I\!N \quad : \quad & \text{there is an isometric embedding} \\ & \Phi : M_d(B(2)) \to M_{d'}(B(q))\}. \end{aligned} \tag{4.105}$$

That means, an isometric embedding $M_d(B(2)) \to M_{d'}(B(q))$ exists if and only if $d' \geq W(d,q)$. The Waring number $W(d,q)$ is well-defined as a consequence of the proof by Hilbert and Stridsberg, compare [227]. Moreover,

Remark 4.16.2 *(Lyubich, Vaserstein [259]) For the Waring numbers the following are known, where q is an even integer:*

[33] Fortunately, there is a well-known mathematical question which needs these maps. The following isometric embeddings $\Phi : M_d(B(2)) \to M_{d'}(B(q))$ are known in connection with Waring's problem, which is a problem in number theory:

J.Liouville:	$d=4$	$d'=12$	$q=4$
E.Lucas:	$d=3$	$d'=7$	$q=4$
A.Fleck:	$d=4$	$d'=32$	$q=6$
A.Hurwitz:	$d=4$	$d'=72$	$q=8$
I.Schur:	$d=4$	$d'=72$	$q=10$

compare [227], [229] and [379]. For a little bit of the history of Waring's problem see [356].

a) $W(d,q)$ *is monotone, which means* $W(d-1,q) \leq W(d,q) \leq W(d,q+2)$.

b) $W(2,q) = q/2 + 1$.

c) $W(d,q)$ *grows exponentially in the dimension:*

$$\binom{d+q/2-1}{d-1} \leq W(d,q) \leq \binom{d+q-1}{d-1}.$$

An exact value of $W(d,q)$ is only known for small values of d and q. König [229], Lyubich, Vaserstein [259] and Seidel [324] reported and computed several Waring numbers exactly:

$$\begin{array}{rcr} W(3,4) & = & 6 \\ W(3,6) & = & 11 \\ W(3,8) & = & 16 \\ W(4,4) & = & 11 \\ W(7,4) & = & 28 \\ W(8,6) & = & 120 \\ W(23,4) & = & 276 \\ W(23,6) & = & 2300 \\ W(24,10) & = & 98280 \end{array}$$

In view of the properties of the Waring number we obtain

Theorem 4.16.3 *(Cieslik [86], [89]) For the Steiner ratio of* $\mathcal{L}_q^{d'}$, *where* q *is an even integer, we have*

$$m(d,2) \geq m(d',q)$$

for any dimension $d' \geq W(d,q)$.

Using our knowledge about the Waring numbers we find the following bounds for the Steiner ratio of finite-dimensional $\mathcal{L}_p$-spaces.

Corollary 4.16.4 *(Cieslik [86])*

a) *The Steiner ratio of* $\mathcal{L}_4^d$ *has the following upper bounds:*

$$\begin{array}{llll} m(d,4) \leq & & 0.79280\ldots & \textit{for } d \geq 2; \\ m(d,4) \leq & m(4,2) \leq & 0.76871\ldots & \textit{for } d > 10; \\ m(d,4) \leq & m(7,2) \leq & 0.72247\ldots & \textit{for } d > 28; \\ m(d,4) \leq & m(23,2) \leq & 0.69839\ldots & \textit{for } d > 275. \end{array}$$

b) *The Steiner ratio of* $\mathcal{L}_6^d$ *has the following upper bounds:*

$$\begin{array}{llll} m(d,6) \leq & m(3,2) \leq & 0.78419\ldots & \textit{for } d > 10; \\ m(d,6) \leq & m(8,2) \leq & 0.69455\ldots & \textit{for } d > 119; \\ m(d,6) \leq & m(23,2) \leq & 0.69839\ldots & \textit{for } d > 2299. \end{array}$$

c) *The Steiner ratio of* $\mathcal{L}_8^d$ *has the following upper bounds:*

$$\begin{array}{llll} m(d,8) \leq & m(3,2) \leq & 0.78419\ldots & \textit{for } d > 15. \end{array}$$

d) *The Steiner ratio of* $\mathcal{L}_{10}^d$ *has the following upper bounds:*
$m(d,10) \leq \quad m(24,2) \leq \quad 0.69839\ldots \quad$ *for* $d > 98279$.

Remember that $m(d,p)$ is a monotonically decreasing sequence in d.

Corollary 4.16.5 *For any even integer* p*:* $\lim_{d\to\infty} m(d,p) \leq 0.66983\ldots$.

Proof. $W(d,q)$ increases in the dimension d, see 4.16.2(c) and (d). Consequently, if the even number q is fixed, then the Steiner ratio $m(d,q)$ tends to a limit less than or equal to the limit of $m(d,2)$ which has been given in 4.12.2

□

4.17 Banach-Minkowski Spaces of high Dimensions

There is a fundamental result of Dvoretzky on almost ellipsoidal sections of convex bodies. First we use the finite-dimensional version.

I. There holds the following counterintuitive geometric assertion: Each unit ball in a sufficiently large dimensional Banach space has a large, almost ellipsoidal section.[34] More exactly, we use the Banach-Mazur distance, which is a natural similarity measure for two Banach spaces of the same dimension, in the following way: Let $\underline{B}_d$ denote the class of all unit balls in A_d, and let $[\underline{B}_d]$ be affine equivalence classes for $\underline{B}_d$. Then the Banach-Mazur distance Δ is a metric on $[\underline{B}_d]$ defined as

$$\begin{aligned}\Delta([B],[B']) \quad = \quad & \ln\inf\{h \geq 1 : \text{there is a bijective linear mapping } \Phi \\ & \text{such that } B \subseteq \Phi B \subseteq hB\} \qquad (4.106)\end{aligned}$$

for $[B],[B'] \in [\underline{B}_d]$.

Remark 4.17.1 *(Dvoretzky [142], [385]) For each positive real number* ϵ *and each positive integer* d' *there is a number* $D(\epsilon,d')$ *such that every Banach-Minkowski space* $M_d(B)$ *of dimension* d *at least* $D(\epsilon,d')$ *contains a* d'*-dimensional subspace* $M_{d'}(B')$ *such that*

$$\Delta([B'],[B(2)]) \leq \ln(1+\epsilon).$$

[262] and [385] give $D(\epsilon,d') = e^{O(d'/\epsilon^2)}$ as the best known estimate.

In terms of norms 4.17.1 means: For every positive integer d' and every positive real number ϵ there exists a number $D(\epsilon,d')$ such that for every norm $||.||$ in A_d, where $d \geq D(\epsilon,d')$, there exists a constant $c > 0$ and a subspace $A_{d'}$ such that

$$c\cdot ||v||_{\tilde{B}} \leq ||v|| \leq (1+\epsilon)\cdot c\cdot ||v||_{\tilde{B}} \qquad (4.107)$$

[34] But compare 4.5.2, which said that each unit ball can approximate from inside and from outside by similar ellipsoids with ratio $1/\sqrt{d}$, when d denotes the dimension.

for all $v \in A_{d'}$, where $M_{d'}(\tilde{B})$ is isometric to the d'-dimensional Euclidean space.

Suppose that the assumption of remark 4.17.1 is satisfied, and $M_{d'}(B')$ is the subspace of $M_d(B)$. Then we have,

$$m_d(B) \leq m_{d'}(B'). \tag{4.108}$$

Moreover, the inequality

$$\Delta([B'],[B(2)]) \leq \ln(1+\epsilon) \tag{4.109}$$

implies (4.107). Then it is not hard to see that

$$m_{d'}(B') \leq (1+\epsilon)\cdot m_{d'}(B(2)). \tag{4.110}$$

Both, (4.108) and (4.110), give the following

Theorem 4.17.2 *(Cieslik [98]) For the positive integer d' and the positive real number ϵ let $D(\epsilon, d')$ be the Dvoretzky number, as defined in 4.17.1. Then for each Banach-Minkowski space $M_d(B)$ of dimension d at least $D(\epsilon, d')$, the inequality*

$$m_d(B) \leq (1+\epsilon)\cdot m_{d'}(B(2))$$

holds.

II. We defined the quantity C_d as the upper bound of the numbers $m_d(B)$ ranging over all unit balls B of the d-dimensional affine space:

$$C_d = \sup\{m_d(B) : B \text{ a unit ball in } A_d\}. \tag{4.111}$$

In view of 4.3.7 and 3.2.1 the sequence $\{C_d\}_{d=1,2,\ldots}$, starts with $C_1 = 1$, and is a decreasing and bounded, consequently a convergent one. 4.17.2 implies

$$m_d(B(2)) \leq C_d \leq (1+\epsilon)\cdot m_{d'}(B(2)) \leq (1+\epsilon)\cdot C_{d'},$$

if $d \geq D(\epsilon, d')$. Suppose that d' runs to infinity, then d does as well. Hence,

Theorem 4.17.3 *(Cieslik [85], [98]) Let the quantity C_d be defined as the upper bound of all numbers $m_d(B)$ ranging over all unit balls B of the d-dimensional affine space. Then $\{C_d\}_{d=1,2,\ldots}$ is a decreasing and convergent sequence with*

$$\lim_{d\to\infty} C_d = \lim_{d\to\infty} m_d(B(2)).$$

In other terms, if the dimension runs to infinity, the conjecture 4.11.3 is true.

On the other hand, we are interested in

$$c_d = \inf\{m_d(B) : B \text{ a unit ball in } A_d\}. \tag{4.112}$$

Using 3.2.1 and 4.3.1 we obtain

Theorem 4.17.4 *Let the quantity c_d be defined as the lower bound of all numbers $m_d(B)$ ranging over all unit balls B of the d-dimensional affine space. Then $\{c_d\}_{d=1,2,\ldots}$ is a convergent sequence with*

$$\lim_{d\to\infty} c_d = \frac{1}{2}.$$

4.18 The Steiner Ratio of dual Spaces

As specification of our considerations above, we find the following results for $\mathcal{L}_4^2$.

$$\sqrt{\frac{3}{8}} \cdot \sqrt{\sqrt{2}} = 0.72823\ldots \leq m(2,4) \tag{4.113}$$

and

$$m(2,4) \leq \frac{2}{3} \cdot \sqrt{\sqrt{2}} = 0.79280\ldots. \tag{4.114}$$

On the other hand, considering the dual plane $\mathcal{L}_{4/3}^2$, which means we look for $m(2,4/3)$, and obtain the same estimates. Is this a general fact?

Conjecture 4.18.1 *(Du et al. [137]) The Steiner ratio in a Banach-Minkowski space equals that in its dual space: $m_d(B) = m_d(DB)$. In particular, $m(d,p) = m(d,q)$ for $1/q + 1/p = 1$.*

Maybe this conjecture is true in the planar case, which is supported by several facts, see [100]. Particularly,

Theorem 4.18.2 *(Wan et al. [364]) The conjecture 4.18.1 is true for sets with at most five points, that is whenever $n \leq 5$,*

$$m_2^n(DB) = m_2^n(B). \tag{4.115}$$

The relation for $d > 2$ is still an open problem; we have another situation, motivated by investigations of $\mathcal{L}_p$-spaces, where in the plane we find similar behavior of the duals, but in higher-dimensional spaces there are several differences, for instance see the facts of the vertex-degrees discussed in [83], [84] or [341].
In 4.3.2 there is conjectured that $m(3,1) = 3/5 = 0.6$, but in 4.14.1 we saw that $m(3,\infty) \leq 4/7 = 0.571\ldots$. And in general,

Theorem 4.18.3 *Consider Banach-Minkowski spaces of a dimension d. For each $d \geq 3$, at least one of the conjectures 4.3.2 and 4.18.1 is false.*

Proof. Assuming that both conjectures are true. Then

$$\frac{d}{2d-1} = m(d,1) = m(d,\infty) \leq \frac{2^{d-1}}{2^d - 1}, \tag{4.116}$$

using 4.14.1. For $d \geq 3$ this is not a correct inequality.

□

I think the different behavior in two and in higher-dimensional spaces is not strange, since there are two classes of objects of importance in convex geometry, lines and hyperplanes. These classes are the same if and only if the dimension of the space equals two.[35]

[35] For more effects when the dimension jumps from two to three see [92]. For instance, the following property of full trees can be empirically observed: The "typical" set of given points in the Euclidean plane usually does not have SMT's which are full trees. That is, its SMT's tend to be unions of small full trees. On the other hand, the "typical" point sets in the Euclidean spaces of dimension at least three usually do have SMT's which are full trees, or at any rate are unions of very large full trees.

Chapter 5

The Steiner Ratio of Banach-Wiener Spaces

A Banach-Wiener space is an infinite-dimensional linear space equipped with a norm, which makes the derived metric space complete. Investigating Steiner's Problem we consider Banach space theory as a branch of the topology/geometry, and in use a little bit of functional analysis.[1]

5.1 Steiners Problem in Banach-Wiener Spaces

Now, we are interested in normed spaces which are not necessarily finite-dimensional. The idea of normed spaces is based on the same assumption of a norm as in the finite-dimensional case, namely that each vector of a space can be assigned its "length", which satisfies some "natural" conditions: positivity, identity, homogeneity and the triangle inequality, written as $||\cdot||$. Then we derive a metric by $\rho(v,w) = ||v-w||$. ρ will be referred to as the metric associated with the norm $||.||$.
When we additionally assume that the space is complete (with respect to ρ), it is usually called a Banach space.[2] We divided normed spaces in infinite- and finite-dimensional ones. Finite-dimensional spaces are called Banach-Minkowski spaces, which are in any case complete, and were discussed in the chapter before. Infinite dimensional ones we investigate now. A complete one is called a Banach-Wiener space.[3]

The class of infinite-dimensional normed spaces is more intrinsically complicated than the one of finite-dimensional ones.[4] Here, we have to define Steiner's Problem

[1] For this approach compare [53]. For more information see [253] or [254].

[2] A space is called complete if every Cauchy sequence in the space is convergent. For instance the set of all continuous functions equipped with supremum norm is complete, but with sum norm not.

[3] For the name compare [375]. For more information about Banach spaces see [18] and [291].

[4] For instance, in finite-dimensional spaces all norms are equivalent. Thus, on a finite-dimensional space there exists only one, namely a "natural" topology, which means that its convergence is the same as the coordinate ones. This is not true in infinite-dimensional spaces. Moreover, in infinite-

more carefully: Since the demand of shortness forces the network to be cycle-less, it is only necessary to consider trees. Let N be a finite set of points in the space X. For a given natural number k and for k points $v_1, ..., v_k \in X \setminus N$, let $T(k, v_1, ..., v_k)$ be a spanning tree of minimal length in the complete graph with the set $N \cup \{v_1, ..., v_k\}$ of vertices, where the length of the graph is induced by the metric.[5]
If there are both a number k' and points $w_1, ..., w_{k'}$ such that the value

$$L(X)(T(k', w_1, ..., w_{k'}))$$

is minimal among all candidates $T(k, v_1, ..., v_k)$, then we call $T(k', w_1, ..., w_{k'})$ a Steiner Minimal Tree (SMT) for N, and the points $w_1, ..., w_{k'}$ are called Steiner points. That means, an SMT for N is an MST for $N \cup Q$, where Q is a set of additional vertices inserted into the metric space in order to achieve a minimal solution. Whereby, in view of 2.5.12, we may assume that Q is a finite set.

There are normed spaces in which an SMT for specific finite sets does not exist. Baronti, Casini and Papini [23] consider $\mathbf{c}_0$, the usual space of (infinite) sequences of reals with supremum-norm. They show that there are three points in $\mathbf{c}_0$ without a Steiner (here a Torricelli) point. Of course, an MST exists in any case. Hence, we define the Steiner ratio more carefully in the following way:

$$m(X) = \inf\left\{\frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \subseteq X \text{ a finite set for which an SMT exits}\right\}. \quad (5.1)$$

Often we will use the following technique to find an upper bound for $m(X)$: We consider a finite set N of points in X where an MST is easy to calculate. Then we guess a tree T interconnecting N using some additional vertices and having a strictly shorter length. Then we estimate the Steiner ratio by

$$m(X) \le \mu(N) \le \frac{L(T)}{L(\text{MST for } N)},$$

which is motivated by the fact that the quantity $\inf\{L(\text{MST for } N \cup Q)\}$ for any finite set Q always exists.

5.2 Upper and Lower Bounds

To find the range of the Steiner ratio, we recall that the proof of 3.2.1 does not use any specific property of the space. In particular, the dimension is without interest.

Theorem 5.2.1 *The Steiner ratio of any normed space is at least* $1/2$.

dimensional vector spaces X we have a rich supply of inequivalent norms, namely $2^{\dim X}$ many, see [238].

[5]Remember that we saw that a Minimum Spanning Tree always exists.

For any finite-dimensional space we conjectured that its Steiner ratio is essentially greater than 1/2. Now we will show that this is not true for infinite-dimensional ones. Consider the set **c** of all bounded sequences $s = a_0, a_1, a_2, \ldots$ with supremum norm

$$||s|| = \sup\{|a_i| : i = 0, 1, \ldots\}. \tag{5.2}$$

Let s_i be the sequence which consists of the real 0, except the ith position where the real 1 is located. Obviously,

$$||s_i - s_j|| = \begin{cases} 1 & : \quad i \neq j \\ 0 & : \quad \text{otherwise} \end{cases}$$

Now, we investigate the set $N = \{s_0, \ldots, s_{n-1}\}$, and find immediately

$$L(\text{MST for } N) = n - 1. \tag{5.3}$$

Consider the sequence $s = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots$ such that

$$||s_i - s|| = \frac{1}{2} \tag{5.4}$$

for all numbers i. Using s as a Steiner point we find a tree interconnecting N with a length $n/2$. Hence,

$$L(\text{SMT for } N) \leq \frac{n}{2}. \tag{5.5}$$

Thus, the Steiner ratio of the space **c** must be less or equal to $n/2(n-1)$, and this for all values of n. For $n \to \infty$, we have

Theorem 5.2.2 $m(\mathbf{c}) = 0.5$.

Consequently, the bound 1/2 is the best possible lower bound over the class of all normed spaces:

$$c_\infty = \inf\{m(X) : X \text{ an infinite-dimensional normed space}\} = \frac{1}{2}. \tag{5.6}$$

On the other hand, for the upper bound of the Steiner ratio we have the conjecture by Du, Lu, Ngo, Pardalos [137] that for the Steiner ratio of any infinite-dimensional Banach space X

$$m(X) \leq \frac{\sqrt{3}}{4 - \sqrt{2}} = 0.66983\ldots. \tag{5.7}$$

We will show that this conjecture is true. But does equality hold? In other terms, we are interested in the quantity

$$C_\infty = \sup\{m(X) : X \text{ a Banach-Wiener space}\}. \tag{5.8}$$

5.3 Isometric Embeddings

I. X' is called a subspace of X if its restriction preserve the distance:

$$||v - v'||_X = ||v - v'||_{X'} \tag{5.9}$$

for all points $v, v' \in X'$. Assume that we know the Steiner ratio of a normed space X', and furthermore we have that X' is a subspace of the normed space X. Then, similar to the proof of 4.3.7, $m(X) \leq m(X')$. This observation is the core of the present section, but in a less weaker form: We consider functions which map the space X' into X, and which preserve the distance between points. Such a map is called an isometric embedding. More exactly, an function ϕ that maps the space X' into a subspace of the space X is called an isometric embedding of X' into X if

$$||\phi(v) - \phi(v')||_X = ||v - v'||_{X'} \tag{5.10}$$

holds for each pair v and v' of points. Obviously, each isometric embedding is an injective function.[6]

Lemma 5.3.1 *(Cieslik, Reisner [102]) Let an isometric embedding from X' into X be given. Then $m(X') \geq m(X)$.*

Proof. Let N be a finite set in X', and let $\phi : X' \to X$ be an isometric embedding. Then $\phi(N)$ is a finite set in X with the following properties:

- $\phi(N)$ is a set of points in the image $\phi(X')$;
- $\phi(N)$ has the same cardinality as N: $|\phi(N)| = |N|$;
- The mutual distances between the points in N and between the corresponding points in $\phi(N)$ are equal.

This implies the following equation:

$$L(X')(\text{MST for } N) = L(X)(\text{MST for } \phi(N)). \tag{5.11}$$

Moreover,

$$\phi(X') \subseteq X. \tag{5.12}$$

It is possible that an SMT for $\phi(N)$ in the space X is shorter than in the subset $\phi(X')$, but in any case

$$L(X')(\text{SMT for } N) \geq L(X)(\text{SMT for } \phi(N)). \tag{5.13}$$

Both, (5.11) and (5.13), imply the assertion for $\Phi(X')$. Then the theorem follows in view of (5.12).

[6] Note an essential difference between isometries and isometric embeddings for Banach spaces. Whereas an isometry is an affine map (theorem of Ulam and Mazur), this must not be true for isometric embeddings [307]: $X' = (\mathbb{R}, |.|)$ and $X = M_2(B(\infty))$. Let ϕ be the mapping from X' into X given by $\phi(x) = (x, \sin x)$. It is easy to verify that ϕ is a nonlinear isometric embedding. A survey about isometric embeddings of infinite-dimensional spaces are given in [291].

□

Theorem 5.3.2 *Let X be a normed space. Then*

$$m(X) \leq \inf\{m(X') : \text{ there is an isometric embedding from } X' \text{ into } X\}. \tag{5.14}$$

II. Now, we use Dvoretzky's theorem in its infinite-dimensional version. We defined the Banach-Mazur distance in (4.53) for isometries of unit balls in Banach-Minkowski spaces. The Banach-Mazur distance between two not necessarily equal-, and moreover not necessarily finite-dimensional, Banach spaces X and Y can be defined more generally by:

$$\Delta(X, Y) = \ln \inf\{||\Phi|| \cdot ||\Phi^{-1}|| : \Phi : X \to Y \text{ an isomorphism}\}. \tag{5.15}$$

4.17.1 can be generalized to the following counterintuitive geometric assertion: Each unit ball in an infinite-dimensional space has an almost ellipsoidal section. More exactly:

Remark 5.3.3 *(Dvoretzky [142]) Every infinite-dimensional Banach space X contains the space $\mathcal{L}_2^d$ almost isometrically, which means, that for every $\epsilon > 0$ and for every d there is a (Euclidean) unit ball $B(2)$ with*

$$\Delta(X, \mathcal{L}_2^d) < \ln(1 + \epsilon). \tag{5.16}$$

In other terms: For every positive integer d and every real number $\epsilon > 0$ there is a map $\Phi : \mathcal{L}_2^d \to X$ such that

$$||v|| \leq ||\Phi v|| \leq (1 + \epsilon) \cdot ||v|| \tag{5.17}$$

Similar to 4.17.2 and in view of 5.3.2 we get that the conjecture (5.7) is true. Moreover, we find a little bit more by using 4.12.7

Theorem 5.3.4 *(Cieslik, Reisner [102]) Let X be a Banach-Wiener space, then*

$$0.5 \leq m(X) \leq \inf\{m(d, 2) : d \text{ positive integer}\} = \lim_{d \to \infty} m(d, 2) \leq 0.66983\ldots,$$

where $m(d, 2)$ denotes the Steiner ratio of the d-dimensional Euclidean space.

5.4 The Steiner Ratio of ℓ_p

For the number $p \geq 1$ consider the set ℓ_p of all (infinite) sequences $s = \{a_k\}_{k=0,1,\ldots}$ where the norm

$$||s|| = \left(\sum_{k=0}^{\infty} |a_k|^p \right)^{1/p}, \tag{5.18}$$

exists. Similarly, we define the space ℓ_∞ of all convergent sequences with the norm

$$||s|| = \sup\{|a_i| : i = 0, \ldots\}. \tag{5.19}$$

ℓ_∞ is a subspace of $\mathbf{c}$.

The space ℓ_p contains each $\mathcal{L}_p^d$ isometrically. Hence, $m(\ell_p) \leq m(d,p)$, for any dimensions d. Thus, $m(\ell_p) \leq \lim_{d\to\infty} m(d,p)$. Therefore, in view of 4.15.6, we find:

Theorem 5.4.1

$$m(\ell_p) \leq \left(\frac{1}{2}\right)^{1/p}. \tag{5.20}$$

For $p = 1$ the bound in 5.4.1 is really tight, since

Corollary 5.4.2 $m(\ell_1) = 0.5$.

The inequality

$$\left(\frac{1}{2}\right)^{1/p} \leq C_\infty, \tag{5.21}$$

is equivalent to

$$p \leq -\frac{\ln 2}{\ln C_\infty}. \tag{5.22}$$

In view of (5.7), this is satisfied if $p \leq 1.7328\ldots$. And really, for the "Euclidean" case we find a better bound than given in 5.4.1 by consideration of 4.16.5, namely that

Corollary 5.4.3 $m(\ell_2) \leq \lim_{d\to\infty} m(d,2) \leq 0.66983\ldots$.

Additionally, with exact the same arguments as in the proof of 5.2.2 it follows

Theorem 5.4.4 $m(\ell_\infty) = 0.5$.

We saw that the Steiner ratio of Banach-Wiener spaces lies between 0.5 and $0.66983\ldots$. The lower bound is sharp, the upper bound is an estimate in the worst case. What is the range of this quantity? What is the relation between the two quantities C_∞ and $\lim_{d\to\infty} C_d$? In view of 5.3.4 and 4.17.3, we know

$$C_\infty \leq \lim_{d\to\infty} C_d = \lim_{d\to\infty} m(d,2). \tag{5.23}$$

Specific questions: Is $m(\ell_p)$ a concave function, with maximum value for $p = 2$? What is the exact value of $m(\ell_2)$?

5.5 Spaces of Functions

A real-valued function f on a set X is simply a rule that assign a real number to each element of X. The set of all functions are written by $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$. For $X = \mathbb{N}$ we get the set of all sequences.

I. The collection of all bounded functions defined on $[0,1]$ forms a vector space $I\!R^{[0,1]}$. For every $f \in I\!R^{[0,1]}$ we define a norm by

$$||f||_\infty = \sup\{|f(x)| : 0 \leq x \leq 1\}. \tag{5.24}$$

This space is denoted by $\mathcal{F}_\infty$.

Observation 5.5.1 $m(\mathcal{F}_\infty)) = 1/2$.

Proof. ℓ_∞ is a subspace of $\mathcal{F}_\infty$. In view of 5.4.4, the assertion.

□

II. To go further we consider spaces of continuous functions from $I\!R^{[0,1]}$. For $1 < p < \infty$ we denote by $\mathcal{F}_p$ such space equipped with the norm

$$||f||_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}, \tag{5.25}$$

provided that the integral exists.
For $p = 1$ this is an example for a non-complete normed space, but $\mathcal{F}_\infty$ is complete. And it is easy to see that $||f||_1 \leq ||f||_\infty$ for any $f \in I\!R^{[0,1]}$.

Theorem 5.5.2 $m(\mathcal{F}_1) = 1/2$.

Proof. For a positive integer n we consider the set $N = \{f_0, \ldots, f_{n-1}\}$ of functions whereby:

$$f_i(x) = \begin{cases} 0 & : \quad 0 \leq x \leq \frac{i}{n} \\ 2nx - 2i & : \quad \frac{i}{n} \leq x \leq \frac{i}{n} + \frac{1}{2n} \\ -2nx + 2i + 2 & : \quad \frac{i}{n} + \frac{1}{2n} \leq x \leq \frac{i+1}{n} \\ 0 & : \quad \frac{i+1}{n} \leq x \leq 1 \end{cases}$$

for $i = 0, \ldots, n-1$. It holds

$$||f_i - f_j||_1 = \begin{cases} \frac{1}{n} & : \quad i \neq j \\ 0 & : \quad \text{otherwise} \end{cases}$$

And

$$||f_i - o||_1 = \frac{1}{2n}, \tag{5.26}$$

when o denotes the zero function. It follows $\mu(N) \leq n/(2n-2)$, and therefore the assertion.

□

III. It is well-known that every subspace X of $\mathcal{F}_2$ is isometric to ℓ_2. Therefore, together with 5.4.3

$$m(\mathcal{F}_2) \leq m(X) \leq m(\ell_2) \leq 0.66983\ldots. \tag{5.27}$$

Levy [250] shows that $\mathcal{F}_p$ is isometric to a subspace of $\mathcal{F}_r$ for $1 \leq r \leq p < 2$. Hence,

$$m(\mathcal{F}_r) \leq m(\mathcal{F}_p) \leq m(\mathcal{F}_2). \tag{5.28}$$

This supports

Conjecture 5.5.3 $C_\infty = m(\ell_2)$.

IV. All spaces where we find the Steiner ratio exactly, this value equals 1/2. Is there a Banach-Wiener space X with $m(X) > 0.5$? The main difficulty is that we have not a complete description of Banach-Wiener spaces.[7]

[7] For instance considering subspaces. The simplest infinite-dimensional Banach spaces are ℓ_p and **c**0. It seems that each space contains one of these specific spaces, but this is not true, since Tsirelson [355] gives a Banach-Wiener space which no contains neither **c** nor ℓ_p, $1 \leq p < \infty$. Hence, the following approach that for any infinite dimensional space X it holds $m(X) = 1/2$ fails: X contain all finite-dimensional spaces X', and then $m(X) = \inf\{m(X')\}$.

Chapter 6

The Steiner Ratio of Metric Spaces (cont.)

After considering many examples of metric spaces, we go back to the general case. For a general introduction into metric spaces compare [228], [237] or [307].

6.1 The Ratio

Note that there are metric spaces in which not every finite set has an SMT: Ivanov et al. [207]: Let X be the set of all positive integers. A metric is defined by

$$\rho(m,n) = \begin{cases} 0 & : \quad m = n \\ \frac{1}{m+n} + 1 & : \quad m \neq n \end{cases}$$

Then, consider the three-element set $N = \{(0,0,0),(0,1,1),(1,0,1)\}$ in the complete metric space

$$(X^3, \tilde{\rho}) = \bigotimes_{i=1}^{3} (X, \rho), \tag{6.1}$$

where

$$\tilde{\rho}((x^1,x^2,x^3),(y^1,y^2,y^3)) = \max\{\rho(x^1,y^1), \rho(x^2,y^2), \rho(x^3,y^3)\}. \tag{6.2}$$

The triangle spanned by N is equilateral, since the length of each of its sides equals 2. Hence, the length of an MST for N is 4.
On the other hand, for any point $q \notin N$ we have $\tilde{\rho}(v,q) > 1$. Therefore, the length of an arbitrary tree constructed for $N \cup \{q\}$ is strictly more than 3. But for $q = (t,t,t), t > 1$, we have

$$\sum_{v \in N} \tilde{\rho}(v,q) = 3 + \frac{3}{t} \to 3$$

when $t \to \infty$. Thus, there does not exist an SMT for N in $(X^3, \tilde{\rho})$.

A complete description of all metric spaces in which Steiner's Problem is solvable is not known, and this situation is unlikely to change, because the class of all metric spaces is too big. So it is necessary to prove the existence of an SMT for each specific metric space independently. In view of this situation, we define the Steiner ratio by

$$m(X) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \text{ a finite set in } X \text{ for which an SMT exists} \right\}.$$

Remember that the Steiner ratio of every metric space obeys $m(X, \rho) \geq \frac{1}{2}$, and this is the best possible bound. Now, we show that the complete interval from 0.5 to 1 is the range for the Steiner ratio of metric spaces.

Theorem 6.1.1 *(Ivanov, Tuzhilin [210])*

a) *For any real number between* 0.5 *and* 1 *there is a metric space with this quantity as the Steiner ratio.*

b) *a) remains true for finite spaces.*

Sketch of the *proof.* Consider the metric space $X = \{x_0, x_1, \ldots, x_k\}$ with

$$\rho(x_i, x_j) = \begin{cases} 0 & : \quad i = j \\ a & : \quad i, j = 0, i \neq j \\ 2 & : \quad \text{otherwise,} \end{cases}$$

where a is a variable real number, but with the following constraints:

1. Since ρ should be a metric, we have $2 \leq a + a$. Hence, $1 \leq a$.
2. An MST for $N = \{x_1, \ldots, x_k\}$ has length $2(k-1)$. A shorter tree is given insofar as the star with center x_0 has length $k \cdot a$. Hence, $k \cdot a < 2(k-1)$, which forces $a \leq 2 - \frac{2}{k}$.

If $k = 1$, then $m = 1$; now assume $k \geq 2$, and let $N \subseteq X$ with $|N| = n \geq 2$.

Case 1: $x_0 \in N$.

$$L(\text{MST for } N) = L(\text{SMT for } N) = a(n-1).$$

Hence, $\mu(N) = 1$.

Case 2: $x_0 \notin N$.

$$\begin{aligned} L(\text{MST for } N) &= 2(n-1) \quad \text{and} \\ L(\text{SMT for } N) &= \min\{2(n-1), an\}. \end{aligned}$$

Hence, $\mu(N) = \min\left\{1, \frac{an}{2(n-1)}\right\}$, such that

$$m(X, \rho) = \min_{2 \leq n \leq k} \min\left\{1, \frac{an}{2(n-1)}\right\} = \min\left\{1, \frac{ak}{2(k-1)}\right\} = \frac{ak}{2(k-1)}.$$

This ratio will be equal to a given number m between 0.5 and 1 if and only if $a = 2m(1 - 1/k)$.

□

Consequently, a) There are many metric spaces, including finite ones, with Steiner ratio 1; and b) There are infinite metric spaces of the Steiner ratio 1/2. But there is not a finite one.

6.2 Preserving Properties

Metric embedding techniques have been widely used in recent years in network design, and we saw more than one times that isometry-like maps are helpful. Now, we discuss this fact in a more general sense.

I Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. We say that a surjective map $\Phi : X \to Y$ is an isometry if Φ preserves the metric, that is

$$\rho_Y(\Phi x, \Phi x') = \rho_X(x, x'), \tag{6.3}$$

for all $x, x' \in X$.
Each isometry must be a bijective function. Consider the inverse function Φ^{-1}:

$$\rho_Y(y, y') = \rho_Y(\Phi\Phi^{-1}y, \Phi\Phi^{-1}y') = \rho_X(\Phi^{-1}y, \Phi^{-1}y').$$

This proves that the inverse of an isometry is also an isometry.[1] Isometry is an equivalence relation in the class of all metric spaces.
The generalization of 4.3.4 is easy to see:

Theorem 6.2.1 *If there exists an isometry between the metric spaces (X, ρ_X) and (Y, ρ_Y), then their Steiner ratio are the same:*

$$m(X, \rho_X) = m(Y, \rho_Y). \tag{6.4}$$

II. In a weaker form we consider functions which map the space X' into X, and which preserve the distance between points. Such a map is called an isometric embedding. Obviously,

Theorem 6.2.2 *If there is an isometric embedding from the metric space (X, ρ_X) into (Y, ρ_Y), then*

$$m(X, \rho_X) \geq m(Y, \rho_Y). \tag{6.5}$$

Corollary 6.2.3 *If for two metric spaces (X, ρ_X) and (Y, ρ_Y) there exist both an isometric embedding from (X, ρ_X) into (Y, ρ_Y) and an isometric embedding from (Y, ρ_Y) into (X, ρ_X), then the Steiner ratios are equal*

$$m(X, \rho_X) = m(Y, \rho_Y). \tag{6.6}$$

[1] Consider $(X, \rho_X) = (Y, \rho_Y)$. The product of two isometries is an isometry as well; and the identity is of course an isometry. Consequently the collection of all isometries of a space onto itself forms a group.

Maybe as an paradox, note that if for two metric spaces (X, ρ_X) and (Y, ρ_Y) there exist both an isometric embedding from (X, ρ_X) into (Y, ρ_Y) and an isometric embedding from (Y, ρ_Y) into (X, ρ_X), there it is not necessary that both space isometric. Example: $X = [0, \infty[$ and $Y = X \cup \{-2\}$ both with equipped with $|.|$. Then $\phi(x) = x$ and $\psi(y) = y + 2$, $\psi(-2) = 0$ are isometric embeddings. They are injective, but not surjective mappings; an isometry cannot exist.
Remember that an isometry is a distance preserving mapping of the whole space, whereas an isometric embedding is only a partial map.[2] Of course, the restriction of an isometry is distance preserving. The converse assertion depends on the extendibility of such an mapping to an isometry. This is not a simple question. Benz [22] show that for all Euclidean spaces the following holds true: Let $\mathcal{G}$ be a subset and let ϕ be a distance-preserving of $\mathcal{G}$ into the space. Then there exists an isometry Φ such $\Phi(x) = \phi(x)$ for all $x \in \mathcal{G}$. He also gives an example for other spaces where this is not true.

III. Let (X, ρ) be a metric space and Y a set. Assume that $\phi : X \to Y$ is a bijection. Then we can transfer the metric ρ on X to a metric ρ' on Y in an obvious way:

$$\rho'(\phi(x), \phi(x')) = \rho(x, x'), \tag{6.7}$$

for $x, x' \in X$.

A metric space (X, ρ) is called complete if every Cauchy sequence in X is convergent in X.[3] The metric space (Y, ρ') is said to be a completion of (X, ρ) if there exists an isometry $\Phi : X \to Y$ such that the image $\Phi(X)$ is dense in Y. Well-known, see [237], [253], [254], that for each metric space there exists a completion.
Let (Y, ρ') be the completion of (X, ρ). Then, on one hand,

$$m(X, \rho) = m(\Phi(X), \rho') \geq m(Y, \rho').$$

On the other hand, the density forces equality.

Theorem 6.2.4 *Let the metric space (Y, ρ') be the completion of the space (X, ρ). Then they have the same Steiner ratio:*

$$m(X, \rho) = m(Y, \rho'). \tag{6.8}$$

IV. Let X be a set equipped with two metrics ρ and ρ'. We said that ρ and ρ' are (metric-) order preserving if

$$\rho(x, x') \leq \rho(y, y') \quad \text{if and only if} \quad \rho'(x, x') \leq \rho'(y, y'), \tag{6.9}$$

for all $x, x'y, y' \in X$.

[2]In pure geometric terms: In a space itself, the concept of isometry captures the idea of "motion" of geometric objects. A figure is a non-empty compact set. Two figures $\mathcal{G}$ and $\mathcal{H}$ are isometric if there is a motion (= isometry) that carries $\mathcal{G}$ onto $\mathcal{H}$. They are congruent if there are a distance preserving mapping that carries $\mathcal{G}$ onto $\mathcal{H}$. Wetzel [373] discuss the equivalence of these terms.

[3]Completeness is not a topological property. That is, there are two equivalent metrics ρ and ρ' on a set X such that (X, ρ) is complete, but (X, ρ') not. For normed spaces this cannot be, [237].

Lemma 6.2.5 *Let (X, ρ) be a metric space and let $\Phi : X \to X$ be an isometry. Then ρ and $\rho \circ \Phi$ are metric order preserving.*

The converse question is not simple, see [21]. For example the metrics ρ and $\rho' = \rho/(1+\rho)$ are metric order preserving, but not isometric, since in general ρ' is in any case bounded.
Remember that to find an MST only uses the mutual distances between the points. Then recall 2.3.1 to find

Theorem 6.2.6 *Let X be a set. Suppose that the metrics ρ and ρ' on X are order preserving. Then for any finite set $N \subseteq X$ an MST for N in (X, ρ) is an MST for N in (X, ρ'), and vice versa.*

V. The following theorem was only proved for the case of Banach-Minkowski spaces, but the proof in the general case of metric spaces is just the same.

Theorem 6.2.7 *Let X be a set, and ρ_1 and ρ_2 be two metrics on X. We assume that for some numbers $c_2 \geq c_1 > 0$ and for arbitrary points x and y from X the following inequality holds:*

$$c_1 \cdot \rho_2(x, y) \leq \rho_1(x, y) \leq c_2 \cdot \rho_2(x, y). \tag{6.10}$$

Then

$$\frac{c_1}{c_2} \cdot m(X, \rho_2) \leq m(X, \rho_1) \leq \frac{c_2}{c_1} \cdot m(X, \rho_2). \tag{6.11}$$

Two metrics which satisfy (6.10) are called equivalent.[4] For instance, in finite-dimensional normed spaces all metrics are equivalent. Contrary, in infinite-dimensional spaces there are many inequivalent metrics.
We know by (4.12) that all norms in a finite-dimensional affine space induce the same topology, the well-known topology with coordinate-wise convergence, which is the topology derived from the Euclidean metric. This implies that each linear mapping from a Banach-Minkowski space into another is continuous, and each finitely dimensional space with a norm must be complete.
Conversely, there is exactly one topology that generates a finite-dimensional linear space to a metric linear space satisfying the separating property by Hausdorff, whereby a topological space is called a Hausdorff space if any two different points lie in suitably chosen disjoint open sets. Moreover, if a normed linear space has the property that the unit ball is compact then the space is finite-dimensional. For a proof of all these facts see [352].

6.3 Subspace Properties

I. Let (X, ρ_X) be a metric space. If $Y \subseteq X$, then the restriction of ρ_X on $Y \times Y$ with

$$\rho_Y(x, y) = \rho_X(x, y) \tag{6.12}$$

[4] 6.10 implies that the collection of open sets of two equivalent metrics induces in a set exactly the same topology.

for all points $x, y \in Y$ is a metric on Y. In what follows we regard (Y, ρ_Y) as a metric space and will call it a subspace of (X, ρ).
Let (X, ρ) be a metric space and $Y \subseteq X$ be some of its subspace. Recall that Kruskal's method, which finds an MST, uses only the mutual distances between the points. Hence, it holds that

$$L(Y, \rho)(\text{MST for } N) = L(X, \rho)(\text{MST for } N)$$

for any finite set N of points in Y. On the other hand, it is possible that an SMT for N in the space (X, ρ) is shorter than in the subspace (Y, ρ). That is

$$L(X, \rho)(\text{SMT for } N) \leq L(Y, \rho)(\text{SMT for } N)$$

for any finite set N of points in Y. So we have:

Theorem 6.3.1 *Let (X, ρ) be a metric space, and $Y \subseteq X$ be some of its subspace. Then*

$$m(Y, \rho) \geq m(X, \rho). \tag{6.13}$$

Consequently: Let (X, ρ) be a metric space. Then

$$m(X, \rho) \leq \inf\{m(Y) : Y \text{ a subspace of } X\}. \tag{6.14}$$

For example considering a three-dimensional space $M_3(C)$ normed by a cylinder C as unit ball. Then $m_3(C) \leq 2/3$.

II. The following proposition is new and needs a proof.

Lemma 6.3.2 *Let $f : X \to Y$ be some mapping of a metric space (X, ρ_X) onto a metric space (Y, ρ_Y). We assume that f does not increase the distances, that is, for arbitrary points x and y from X the following inequality holds:*

$$\rho_Y(f(x), f(y)) \leq \rho_X(x, y). \tag{6.15}$$

Then for an arbitrary finite set $N \subseteq Y$ we have:

$$L(X, \rho_X)(\mathit{MST\ for\ } N) \geq L(Y, \rho_Y)(\mathit{MST\ for\ } f(N)) \text{ and} \tag{6.16}$$

$$L(X, \rho_X)(\mathit{SMT\ for\ } N) \geq L(Y, \rho_Y)(\mathit{SMT\ for\ } f(N)). \tag{6.17}$$

Proof. Let G be an arbitrary connected graph constructed on N. We consider length-functions on G defined on the edges $\underline{xy}$ of G as follows:

$$\omega_Y(x, y) = \rho_Y(f(x), f(y)).$$

Since f does not increase the distances, it holds

$$L(X)(G) \geq \omega_Y(G).$$

Let G' be a graph on $N' = f(N)$ such that the number of edges joining the vertices x' and y' from $N' = V(G')$ is equal to the number of edges from G joining the vertices from $f^{-1}(x') \cap N$ with the vertices from $f^{-1}(y') \cap N$. It is clear that G' is connected, and

$$L(Y)(G') = \omega_Y(G).$$

Conversely, it is easy to see that for an arbitrary connected graph G' constructed on $f(N)$ there exists a connected graph G_X on N, such that

$$L(Y)(G') = \omega_Y(G_X).$$

To construct G_X, it suffices to span each set $N \cap f^{-1}(x')$, $x' \in N'$, by a connected graph and then to join each pair of the constructed graphs corresponding to some adjacent vertices in G' by k edges, where k is the multiplicity of the corresponding edge in G'. Therefore,

$$\begin{aligned} L(X)(\text{MST for } N) &= \inf\{L(X)(G) : V(G) = N\} \\ &\geq \inf\{\omega_Y(G) : V(G) = N\} \\ &= \inf\{L(Y)(G') : V(G') = f(N)\} \\ &= L(Y)(\text{MST for } f(N)). \end{aligned}$$

Thereby, the first inequality is proved. Now let us prove the second inequality. We have:

$$\begin{aligned} L(X)(\text{SMT for } N) &= \inf\{L(X)(\text{MST for } \tilde{N}) : \tilde{N} \supset N\} \\ &\geq \inf\{L(Y)(\text{MST for } f(\tilde{N})) : \tilde{N} \supset N\} \\ &\geq \inf\{L(Y)(\text{MST for } \tilde{N'}) : \tilde{N'} \supset f(N)\} \\ &= L(Y)(\text{SMT for } f(N)). \end{aligned}$$

□

This lemma gives two theorems:

Theorem 6.3.3 *(Ivanov, Tuzhilin, Cieslik [212]) Let $f : X \to Y$ be a mapping of a metric space (X, ρ_X) to a metric space (Y, ρ_Y), and let f do not increase the distances. We assume that for each finite subset $N' \subseteq Y$ there exists a finite subset $N \subseteq X$ such that $f(N) = N'$ and*

$$L(X, \rho_X)(SMT \text{ for } N) \leq L(Y, \rho_Y)(SMT \text{ for } N'). \tag{6.18}$$

Then

$$m(X, \rho_X) \leq m(Y, \rho_Y). \tag{6.19}$$

Theorem 6.3.3 can be slightly reinforced as follows.

Theorem 6.3.4 *(Ivanov, Tuzhilin, Cieslik [212]) Let $f : X \to Y$ be a mapping of a metric space (X, ρ_X) to a metric space (Y, ρ_Y), and let f do not increase the distances. We assume that for each finite subset $N' \subseteq Y$ the following inequality holds:*

$$\inf\{L(X,\rho_X)(SMT\ for\ N) : f(N) = N'\} \leq L(Y,\rho_Y)(SMT\ for\ N'). \tag{6.20}$$

Then

$$m(X,\rho_X) \leq m(Y,\rho_Y). \tag{6.21}$$

6.4 The Steiner Ratio of ultrametric Spaces

Up to now, we have found in each space that the determination of an SMT is a hard problem. In the next example, we describe a class of metric spaces in which Steiner's Problem is as easy as finding a Minimum Spanning Tree.

Let (X, ρ) be a metric space. ρ is called an ultrametric if for any points x, y, z in X

$$\rho(x,y) \leq \max\{\rho(x,z), \rho(y,z)\} \tag{6.22}$$

For each set X there is an ultrametric by setting $\rho(x, y) = 1$ for any different $x, y \in X$. Another, not so simple, example is given for the set $I\!N$ of all nonnegative integers, and $\rho(m, n) = \max\{1 + \frac{1}{m}, 1 + \frac{1}{n}\}$ for $m \neq n$. And most popular: For the sequences $x = \{x_i\}, y = \{y_i\} \in I\!N^{I\!N}$ we wish to regard x and y to be close to each other if their first n terms are equal for some large n. This is achieved by the following function:

$$\rho(x,y) = \begin{cases} 0 & : \quad x = y \\ \frac{1}{n} & : \quad \text{otherwise, where } n = \min\{n : x_i \neq y_i\} \end{cases}$$

which creates a (complete) ultrametric.

It is not hard to see that we have

Lemma 6.4.1 *For any three points x, y and z with $\rho(x, z) \neq \rho(y, z)$ in an ultrametric space (X, ρ) it holds*

$$\rho(x,y) = \max\{\rho(x,z), \rho(y,z)\}.$$

That means that all triangles in (X, ρ) are isosceles triangles where the base is the shorter side.

Let $T = (V, E)$ be an SMT for N. If $V \setminus N \neq \emptyset$, then there is a Steiner point q such which is adjacent to two vertices v and v' in N, compare 2.5.5. Using 6.4.1, we may assume that $\rho(v, v') = \rho(v, q)$. The tree $T' = (V, E \setminus \{\underline{vq}\} \cup \{\underline{vv'}\})$ has the same length as T, and it is an SMT for N, too. If $g_{T'}(q) \geq 3$, we repeat this procedure. If $g_{T'}(q) = 2$, we find an SMT with a smaller number of Steiner points than T, since no Steiner point has degree smaller than 2.
Hence, we proved that Steiner's Problem in an ultrametric space is the same as finding an MST. Consequently,

Theorem 6.4.2 *The Steiner ratio of an ultrametric space equals one.*

Knowing that a ultrametric space is "tree-like", which is used in the theory of phylogenetic spaces [198], this result is not a surprise.
The converse statement is not true, since the real line has the Steiner ratio 1, but is not an ultrametric space. But, this is not a good counterexample, since there is *a priori* no Steiner point.
An interesting question: What does the equality $m(X, \rho) = 1$ for a metric space (X, ρ) mean?[5]

[5] It seems, that each metric space with Steiner ratio equals 1 is in a general sense "tree-like".

Chapter 7

The Steiner Ratio of Discrete Metric Spaces

We consider metric spaces, defined by the property that each bounded set is a finite one.[1] Since for a given set of points the set of Steiner points is a bounded one, we may assume that for any finite set an SMT exists; the Steiner ratio is well-defined.

7.1 The Steiner Ratio of Graphs

A network is a (connected) graph $G = (V, E)$ equipped with a length-function $f : E \to I\!R$.
A network is a metric space (V, ρ) by defining the distance function in the way that $\rho(v, v')$ is the length of a shortest path between the vertices v and v' in G. If there does not exist a length-function explicitly, we assume $f \equiv 1$, that means the distance $\rho(v, v')$ is defined as the minimal number of edges connecting the vertices v and v' by a path in G. A survey about graphs as metric spaces is presented in [387].
In this sense, we construct the so-called metric closure G^f defined as the complete graph on V such that the length of an edge $\underline{vv'}$ in G^f is the length of a shortest path between v and v' in G.[2,3]

[1] The term "discrete metric space" has sometimes another meaning, compare [284] and [303].

[2] Using Dijkstra's algorithm [114] a shortest path can be found in polynomially bounded time. The algorithm is a consequence of the following principle, which is the mother of dynamic programming:

Observation 7.1.1 *(Bellman [24]) Let $G = (V, E)$ be a graph, and let v and v' be two vertices of G. If $e = \underline{wv'}$ is the final edge of some shortest path $v, \ldots, w, v'$ from v to v', then $v, \ldots, w$ (that is the path without the edge e) is a shortest path from v to w.*

[3] When we are only interested in the metric ρ for G^f, we can find the metric closure in a simpler way:

Algorithm 7.1.2 *(Floyd [150]) Let $G = (V = \{v_1, \ldots, v_n\}, E, f)$ be a network. The metric closure $G^f = (V, \rho)$ can be found by the following procedure:*

On the other hand, each finite metric space is a desired chosen finite graph:

Observation 7.1.3 *(Hakimi, Yau [183]) Each finite metric space can be represented as a finite graph with a (nonnegative) length-function.*

Proof. Let (X, ρ) be a finite metric space. We define the graph $G = (X, E)$ as the complete graph on the vertex-set X. The length-function f is given by the metric ρ.

□

Now, Steiner's Problem in Graphs is to find for a connected graph $G = (V, E)$ with a length-function $f : E \to \mathbb{R}$, and a nonempty subset N of V, a connected subgraph $G' = (V', E')$ of G with $N \subseteq V'$ such that

$$L(G') = \sum_{e \in E'} f(e) \tag{7.1}$$

is minimal.[4]

Steiner's Problem in graphs was originally formulated by Hakimi [184] in 1971. Since then, the problem has received considerable attention in the literature. A collection of equivalent formulations for Steiner's Problem in graphs is given in [225]; solution methods in [95], [118], [138], [184], [202], [301], [361], [367] and [376].[5]

It should note that Steiner's Problem in graphs is $\mathcal{NP}$-hard, [221], and this remains true if any one of the following conditions hold: All edge lengths are equal, i.e. the

---

1. *for $\underline{vv'} \notin E$ define $f(\underline{vv'}) = \infty$;*
2. *for $i := 1$ to n do for $j := 1$ to n do $\rho(v_i, v_j) := f(\underline{v_i v_j})$;*
3. *for $i := 1$ to n do for $j := 1$ to n do for $k := 1$ to n do*
   if $\rho(v_j, v_i) + \rho(v_i, v_k) < \rho(v_j, v_k)$ then $\rho(v_j, v_k) := \rho(v_j, v_i) + \rho(v_i, v_k)$.

A complete discussion of metric closures are given in [197].

[4]This is a common generalization of two well-known problems in network design:

$|N| = 2$: We search a shortest path interconnecting two vertices in the graph G.

$N = V$: We look for a minimum spanning tree in G.

Therefore we may assume that for $n = |N|$ not far from 2 or $|V|$, respectively, there are methods for finding an SMT quickly.

[5]A solution technique uses 2.5.6 and 2.5.12:

Algorithm 7.1.4 *(Hakimi [184], Lawler [242]) Let $G = (V, E, f)$ be a network. Let $N \subseteq V$ be a set of given points. Then an SMT for N in G can found by the following procedure:*

1. *Compute shortest paths between all pairs of vertices;*
   Replace the edge lengths with the shortest path lengths, adding edges to the graph where necessary;
2. *For each possible subset $V' \subseteq V \setminus N$ such that $0 \leq |V'| \leq |N| - 2$, find a minimum spanning tree $T(N \cup V')$ in the induced subgraph $G^f[N \cup V']$;*
3. *Select the shortest spanning tree from the ones computed in step 2;*
   Transform it into a tree of the original graph, i.e., replace each edge of the spanning tree with the edges of the shortest path between the vertices.

length of a subgraph is its number of edges [221]; the graph is bipartite [162]; the graph is a hypercube [151]; the graph G is planar [162], [301]; or the graph is a grid [163].[6] An annotated bibliography is presented in [304].

Now, the Steiner ratio in graphs is of the form

$$m = m(G) = \min\left\{\frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \subseteq V\right\}. \tag{7.2}$$

In other words, $m = m(G^f) = m(V, \rho)$, where G^f denotes the metric closure of the the graph G with length-function f.

Theorem 7.1.5 *For the Steiner ratios of (connected) graphs it holds*

a) *The Steiner ratio of complete graphs, paths and cycles equals 1.*

b) *Let G be a star with k leaves, $k \geq 2$. Then $m(G) = k/(2k-2)$.*

Proof. a) is obvious.
b) Let S_k be a star with k leaves which are the given points, we find an MST of length $2 \cdot (k-1)$ and an SMT of length k. Hence, It is easy to see that all other sets of given points do not give a smaller value of the Steiner ratio.

□

Corollary 7.1.6 *Let G be a (connected) graph. Then for the Steiner ratio of G $\frac{1}{2} \leq m(G) \leq 1$. These bounds are the best possible ones.*

All our considerations immediately suggests an approximation algorithm for Steiner's Problem in graphs:

Algorithm 7.1.7 *(Kou, Markowsky, Berman [232]) A finite set N of n vertices in a network $G = (V, E, f)$ is given. Then*

1. *Describe the metric closure $G^f = (V, \rho)$ and for all $v, v' \in N, v \neq v'$ and the shortest paths $G(v, \ldots, v')$;*
2. *Find an MST $T = (N, F)$ for N in (V, ρ);*
   Set $F' := \bigcup_{\underline{vv'} \in F} G(v, \ldots, v')$; and
   Set $V' := \bigcup_{\underline{vv'} \in F'} \{v, v'\}$;
3. *While there is a cycle C in (V', F') delete any edge from C;*
   Delete leaves which are not members of N.

It is easy to see that this algorithm is a $1/m(G^f) \leq 2$-approximation algorithm for Steiner's Problem in graphs; and runs in cubic time. For methods to improve the running time of 7.1.7 see [173], [264] and [301]. [295] presents a modification with the help of a preliminary increase by adding Steiner points and compare in [296] it with other methods.

[6] Only in some specific cases Steiner's Problem can be solved in polynomially bounded time: [1], [33], [50], [94], [107], [202], [302], [362] and [376]. Polynomial algorithms are proposed by [33] and [302] for graphs with all given points belonging to a fixed number of faces. In particular this is given for outer-planar graphs.

7.2 Finite Metric Spaces

I. Steiner's Problem in networks and in finite metric spaces are essentially the same. This is a consequence of 7.1.3 and can be seen as follows: First, a solution $G' = (V', E')$ of Steiner's Problem must be a tree, because it is connected and acyclic. Consider the vertices in $V' \setminus N$. Such vertices v with $g_{G'}(v) \geq 3$ are Steiner points, and the vertices v with $g_{G'}(v) = 2$ lie on a shortest path between Steiner points and given points of N. In other terms, we consider Steiner's Problem in the metric closure G^f. The length of the SMT in both graphs must be the same.[7]

II. Let us consider metric spaces with a small number of points. For two- and three-point sets X we have $m(X, \rho) = 1$. An immediate consequence of 3.2.4 is

Theorem 7.2.2 *Let (X, ρ) be a metric space with four points. Then*

$$m(X, \rho) \geq \frac{3}{4}. \tag{7.3}$$

III. Up to now we discussed extensively normed spaces and saw that isometric embeddings can be helpful. Now we bring together both investigations. Although the spaces $\mathcal{L}_p^d$ and ℓ_p are interesting mathematical objects and considered on its own, we also investigated embeddability into these spaces as a tool for estimating the Steiner ratio of finite metric spaces.

For a finite set X the space $\mathbb{R}^X$ is a $|X|$-dimensional affine space. Furthermore,

Lemma 7.2.3 *(Frechet's lemma, compare [262]) Any finite metric space (X, ρ) can isometrically embedded in $\mathcal{L}_\infty^{|X|}$.*[8]

Sketch of the *proof.* The coordinates in ℓ_∞ are indexed by the points of X. The xth coordinate is given by $\phi_x(y) = \rho(x, y)$. The embedding ϕ is not expanding by the triangle inequality On the other hand,

$$||\phi(x) - \phi(y)||_{B(\infty)} = |\phi_x(y) - \phi_y(x)| = \rho(x, y).$$

$\square$

[7] In particular, there is no loss of generality in requiring that the length function satisfy the triangle inequality; if it does not, construct the metric closure.
When we restrict to metric spaces with integer-valued distances we have a deeper result.

Observation 7.2.1 *(Kay, Chartrand [224]) Let (X, ρ) be a finite metric spaces where all values of ρ are integers. Then (X, ρ) is a distance space of some graph if and only if for any two points u and v with $\rho(u, v) \geq 2$ there is a third point w such that*

$$\rho(u, w) + \rho(w, v) = \rho(u, v).$$

[8] And consequently, each finite metric space embeds isometrically into ℓ_∞.

Consequently,

$$m(X,\rho) \geq m(|X|,\infty). \tag{7.4}$$

We discussed the Steiner ratio of $\mathcal{L}_\infty^{|X|}$ in 4.14.1. Therefore, we find again that no finite metric space has Steiner ratio 1/2:

Theorem 7.2.4 *We assume that conjecture 4.14.2 is true. For any finite metric space (X,ρ) it holds*

$$m(X,\rho) \geq \frac{2^{|X|-1}}{2^{|X|}-1}. \tag{7.5}$$

IV. Furthermore we are interested of isometrically embedding finite metric spaces (X,ρ) in Euclidean spaces $\mathcal{L}_2^d$, such that $m(X,\rho) \geq m(d,2)$.
It is obviously evident that any two-point space embeds in the real line $(I\!R, |.|)$). In view of the triangle inequality any three-point metric space embeds isometrically in the Euclidean plane. But for the Steiner ratio this is without interest, since such a space has ratio 1.
But unfortunately we cannot go further. There exist are four-point metric spaces that admit no isometric embedding into the three-dimensional Euclidean space or indeed into Euclidean spaces of any dimension. Robinson [306] construct such a space in the by $X = \{x_1,\ldots,x_4\}$ with $\rho(x_1,x_2) = \rho(x_1,x_3) = \rho(x_2,x_3) = \rho(x_1,x_4) = 2l$ and $\rho(x_2,x_4) = \rho(x_3,x_4) = l$. The triangle inequality is valid for (X,ρ). Any isometric embedding $\phi : X \to I\!R^d$ necessarily maps the point x_4 to the midpoint y of the line joining $\phi(x_2)$ and $\phi(x_3)$. As the Euclidean distance between $\phi(x_1)$ and y equals $\sqrt{3}\cdot l$ forces $\rho(x_1,x_4) = \sqrt{3}\cdot l$ as well.

V. Assume that a finite metric space (X,ρ) is a subspace of ℓ_p, then (X,ρ) can be isometrically embedded into $\mathcal{L}_p^d$ with $d \leq \binom{n}{2}$, $n = |X|$, [262]. With help of 6.2.2 we obtain $m(X,\rho) \geq m(d,p) \geq m(\binom{n}{2},p)$.

Theorem 7.2.5 *If a finite metric space (X,ρ) is a subspace of ℓ_p. Then*

$$m(X,\rho) \geq m(\binom{|X|}{2},p). \tag{7.6}$$

7.3 The Steiner Ratio of Hamming spaces

I. For a word $v \in \{0,1\}^d$ we define the Hamming weight $\mathrm{wt}(v)$ as the number of times the digit "1" occurs in v. For two words v and w over $\{0,1\}$ the Hamming distance is defined by

$$\rho_H(v,w) = \mathrm{wt}(v+w) = \mathrm{wt}(v-w). \tag{7.7}$$

Conversely, $\mathrm{wt}(v) = \rho_H(v,o)$, where $o = 0^d$. In other terms: The Hamming distance between v and w is the number of positions in which v and w disagree. $Q^d = (\{0,1\}^d,\rho_H)$ is called the d-dimensional hypercube. Q^d is a finite metric space with 2^d points.

An interesting observation: Let $\mathcal{L}_1^d$ be the d-dimensional affine space with rectilinear distance. It is easy to see that Q^d can be represented by a d-dimensional rectilinear network, which means that each vertex is identified with a point of the space and two vertices are connected by an edge if and only if the corresponding points differ in exactly one coordinate. In other terms, Q^d is a subspace of $\mathcal{L}_1^d$. Hence, by 6.3.1:

$$m(\mathcal{L}_1^d) \leq m(Q^d) \leq \frac{d}{2(d-1)}.$$

On the other hand, we saw a stronger result in

$$m(\mathcal{L}_1^d) \leq \frac{d}{2d-1}.$$

Miller and Perkel [269] give several results for Steiner's Problem in hypercubes.[9]

II. Now we investigate the quantities $m^n(Q^d)$, assuming that $n \leq d$. In view of 3.2.4 we have

Theorem 7.3.1

$$m^3(Q^d) = \frac{3}{4}. \tag{7.10}$$

The following generalization is simple to see.

Theorem 7.3.2 *Assuming $n \leq d$. Then*

$$m^n(Q^d) \leq \frac{n}{2(n-1)}. \tag{7.11}$$

III. It can be directly generalized to words over a finite nonempty set A of letters, called an alphabet:

$$\rho_H((a_1, \ldots, a_d), (b_1, \ldots, b_d)) = |\{i : a_i \neq b_i \text{ for } i = 1, \ldots, d\}|, \tag{7.12}$$

whereby d is a nonnegative integer and $a_i, b_i \in A$ for $i = 1, \ldots, d$.
The space (A^d, ρ_H) describes the term "maximum parsimony" for phylogenetic trees exactly [178]: The maximum parsimony problem on character dates is Steiner's Problem in hypercubes. In this sense the construction of SMT's plays an important role. But Steiner's Problem is $\mathcal{NP}$-hard, [151]. Hence, approximations are of interest. Foulds [152] says that if $d \gg 1$, we have $m(A^d, \rho_H) \approx \frac{1}{2}$ based on the following result, which is not hard to prove.

[9] Including exact results for all given sets N with $n = |N| \leq 5$: Let

$$L(d,k) = \max\{L(Q^d)(\text{SMT for } N) : N \subseteq Q^d, |N| = k\}. \tag{7.8}$$

Then, of course, $L(d,k) \leq 2^d - 1$ for any positive integer k, and $L(d,2) = d$. Furthermore, $L(d,3) = d$; $L(d,4) = \lfloor \frac{5d}{3} \rfloor$; and, asymptotically,

$$L(d, k+1) \leq \binom{d}{k+1} + (2 + o(1)) \cdot \frac{\log k}{k} \cdot \binom{d}{k}. \tag{7.9}$$

Theorem 7.3.3

$$\frac{1}{2} \leq m(A^d, \rho_H) \leq \frac{d}{2(d-1)}. \tag{7.13}$$

IV. Let X be a (nonempty) finite set. We define a metric ρ_Δ for the collection of subsets of X by

$$\rho_\Delta(S, S') = |S \Delta S'|, \tag{7.14}$$

where Δ denotes the symmetric difference of sets:

$$S \triangle S' = (S \cup S') \setminus (S \cap S') = (S \setminus S') \cup (S' \setminus S). \tag{7.15}$$

Consequently,

$$\rho_\Delta(S, S') = |S \triangle S'| = |S \cup S'| - |S' \cap S| = |S| + |S'| - 2|S' \cap S|.$$

In particular, $\rho_\Delta(S, \emptyset) = |S|$. ρ_Δ represents an analogue of the Hamming distance, since it counts the number of different elements in both sets.[10]

Theorem 7.3.4 *Let X be a finite set with $|X| > 1$. Then*

$$m(\mathcal{P}(X), \rho_\Delta) \leq \frac{|X|}{2(|X|-1)}.$$

Proof. Let $n = |X|$, and consider by S_i, $i = 1, \ldots, n$, the one-element subsets of X. Then $\rho_\Delta(S_i, S_j) = 2$ for $i \neq j$. Hence, an MST for $N = \{S_1, \ldots, S_n\}$ has length $2 \cdot (n-1)$. Using the Steiner point $S = \emptyset$, we have $\rho_\Delta(S_i, S) = 1$, and an SMT has length n: $\mu(N) = n/2(n-1)$.

□

7.4 The Steiner Ratio of Phylogenetic Spaces

We determine the Steiner ratio of phylogenetic spaces.[11] Consider an alphabet A with at least two letters a and b. $A^\star$ denotes the set of all words over A. We define the Levenshtein (or edit) distance, between two words of not necessarily equal length by the minimal number of "edit operations" required to change one word into the other,

[10] This concept can be extended to sets in general by setting $\rho_\Delta(S, S') = \text{volume}(S \triangle S')$. This metric induces the same topology as the Hausdorff-distance.

[11] Phylogenetic spaces play an important role in determining a phylogenetic (evolutionary) tree for a given set of species. Minimum-length trees are widely used for estimating phylogenetic relationships from aligned sequence data, [117]. The minimum-length method selects the trees which require the fewest evolutionary steps on its edges to account for the variation within the characters [19]. Fitch [149] first defined the course of evolution in this sense and suggested an algorithm for solving this problem. For an introduction to Computational Molecular Biology including these aspects see [105], [178], [325], [326] and [368].
The most problem in morphological phylogenetics is to select the characters. Moreover, it must be coded if there are more than two distinct possibilities. For the deep interrelation of this character-based phylogeny approach and Steiner's Problem compare Fernández-Baca [148].

where an edit operation is a deletion, insertion, or substitution of a single letter in either word.

At first glance, it seems that the sequence spaces are subspaces of the phylogenetic space, but this is not true: Consider the two words $v = (ab)^d$ and $w = (ba)^d$; then $\rho_L(v, w) = 2$, but $\rho_H(v, w) = 2d$.
To extend the Hamming distance to a metric for all words, we may use the following way: Let A be a set of letters. Add a "dummy" letter "-" to A. We define a map

$$cl : (A \cup \{-\})^\star \to A^\star \tag{7.16}$$

deleting all dummies in a word from $(A \cup \{-\})^\star$. Then for two words w and w' in $A^\star$ we define the extended Hamming-distance as

$$\begin{aligned} \rho(w, w') \quad &= \min\{\rho_H(\underline{w}, \underline{w'}) : \quad \underline{w}, \underline{w'} \in (A \cup \{-\})^\star, |\underline{w}| = |\underline{w'}|, \\ &\qquad cl(\underline{w}) = w, cl(\underline{w'}) = w'\}. \end{aligned} \tag{7.17}$$

The extended Hamming distance coincides with the Levenshtein metric.
For a generalization of the Levenshtein distance see [100]. Techniques to find an SMT in phylogenetic spaces are described in [96] and [317].

w_i is a word which consists of the letter a repeated d times, except the i-th position where another letter b is located, $i = 1, \ldots, d$. Then consider the set

$$N(d) = \{w_i : |w_i| = d, i = 1, \ldots, d\} \tag{7.18}$$

of d words. For $i \neq j$ it holds that $\rho_L(w_i, w_j) = 2$. Hence,

$$L(\text{MST for } N(d)) = 2(d-1). \tag{7.19}$$

The word $w = a \ldots a$ has distance 1 to any w_i. Consequently, the star with the center w and the leaves w_i, $i = 1, \ldots, d$ is an SMT for $N(d)$, such that

$$L(\text{SMT for } N(d)) = d. \tag{7.20}$$

Both equations (7.19) and (7.20) give $\mu(N(d)) = \frac{d}{2(d-1)}$ for $d \geq 2$.

Theorem 7.4.1 *(Cieslik [97], Foulds [152]) For the Steiner ratio of the phylogenetic space $(A^\star, \rho_L)$, $|A| \geq 2$, it holds that*

$$m(A^\star, \rho_L) = \frac{1}{2}. \tag{7.21}$$

Note that we don't have a finite set N_0 of points which achieves the Steiner ratio, that is $\mu(N_0) = 0.5$. And, in view of 3.3.3, we cannot find such set.

Chapter 8

The Steiner Ratio of Manifolds

We are interested in the Steiner ratio of manifolds.[1]

8.1 The Steiner Ratio on Spheres

I. Network minimization problems on the sphere are the so-called Large Region Location Problems. That means, let X be the surface of a Euclidean ball. A metric on X is given by the shortest great circle distance between the points.
Brazil et al. [43] describe several properties of SMT's on the sphere. For instance: The edges consists of the arcs of great circles; Steiner points are of degree three, where the three edges meet at 120^o angles.[2] A general solution method for Steiner's Problem is still unknown except for some special cases, see [43], [251] and [312].[3]

Theorem 8.1.1 *(Rubinstein, Weng [312]) The Steiner ratio for spheres is the same as in the Euclidean plane.*

Ideas of a *proof.*
a) Suppose that $\triangle u_1v_1w_1$ and $\triangle u_2v_2w_2$ are two triangles of equal side lengths lying on a sphere X_i, $i = 1, 2$ with radii $r_1 < r_2$ respectively. Then it will prove the existence of a map

$$h : \triangle u_1v_1w_1 \to \triangle u_2v_2w_2 \tag{8.1}$$

such that for any two points $p_1, q_1 \in \triangle u_1v_1w_1$ it holds that

$$\rho(p_1, q_1) \geq \rho(h(p_1), h(q_1)). \tag{8.2}$$

[1] For the theory of manifolds, compare [278] and [338].
[2] And forthcoming then see our observations in the preceding chapter.
[3] A nice application was the construction of a new transpacific fiber-optic trunk, see [348].

Moreover, if p_1 and q_1 are not on the same side, then the inequality is strict. This compression theorem can be applied to compare the minimum of a variable in triangles on two spheres. Then the above assertion follows.
It seems that this proof given in [312] needs similar methods like the proof of the Gilbert-Pollak conjecture given by 4.6.1; and consequently it does create the same gap.
b) But if a) fails, this is not important, since another proof has been given independently in [211], [212], compare 8.3.4.

□

To generalize this result first we know the well-known fact that "The Sphere is not Flat": Robinson [306]) shows that there is no isometric embedding from any nonempty open subset of the sphere into any Euclidean space. But this is not an argument against

Conjecture 8.1.2 *The Steiner ratio of the d-dimensional sphere equals the Steiner ratio of the d-dimensional Euclidean space.*

II. We will use another approach in higher dimension: Consider the sphere

$$S^d = \left\{ (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i^2 = 1/4 \right\}. \tag{8.3}$$

It is not possible to map the whole sphere bijectively and continuously onto the space $\mathbb{R}^d$. However, the stereographic projection maps all but one point: More exactly, let $p = (0, \ldots, 0, 1)$ be the "North pole" and $p' = (0, \ldots, 0, 0)$ be the "South pole" of S^d. The stereographic projection is to map from $S^d \setminus \{p\}$ to $\mathbb{R}^d$, by the way that from p there is a line through a point $x \in S^d \setminus \{p\}$ intersecting $\mathbb{R}^d$.[4] The both spaces $\mathbb{R}^d$ and $S^d \setminus \{p\}$ are homeomorphic.
Adding a point to $S^d \setminus \{p\}$, we complete the space. As an example consider the 2-dimensional case $\phi : S^2 \to \mathbb{R}^2 \cup \{x_\infty\}$.

$$\phi(x) = \begin{cases} \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right) & : \ x \neq (0,0,1) \\ x_\infty & : \ x = (0,0,1) \end{cases}$$

for $x = (x_1, x_2, x_3) \in S^2$. The inverse function ϕ^{-1} is given by

$$\phi^{-1}(y) = \begin{cases} \left(\frac{y_1}{1+y_1^2+y_2^2}, \frac{y_2}{1+y_1^2+y_2^2}, \frac{y_1^2+y_2^2}{1+y_1^2+y_2^2} \right) & : \ y \neq x_\infty \\ (0,0,1) & : \ \text{otherwise} \end{cases}$$

for $y = (y_1, y_2) \in \mathbb{R}^2$.

[4] Conversely, when we add an infinite point x_∞ to $\mathbb{R}^d$, we create a new space.

Stereographic projection is obviously not an isometry, since distances are bounded on the sphere, but unbounded in the plane.

Using the projection we can transfer the metric of the sphere to the (extended) set of all complex numbers z and z':

$$\rho(z,z') = \begin{cases} \frac{|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}} & : \quad z' \neq z_\infty \\ \frac{1}{\sqrt{1+|z|^2}} & : \quad z' = z_\infty \end{cases}$$

ρ is called the chordal metric.

Theorem 8.1.3 *The Steiner ratio of the the space of complex numbers equipped with the chordal metric equals the Steiner ratio of the two-dimensional sphere.*

8.2 Riemannian Metrics

Let M be an arbitrary connected d-dimensional Riemannian manifold. For each piecewise-smooth curve γ we denote by length(γ) the length of γ with respect to the Riemannian metric. By ρ we denote the intrinsic metric generated by the Riemannian metric. Recall that

$$\rho(x,y) = \inf_{\gamma} \mathrm{length}(\gamma), \tag{8.4}$$

where the greatest lower bound is taken over all piecewise-smooth curves γ joining the points x and y.
Let p be a point from M. We consider the normal coordinates $(x^1, \ldots, x^d)$ centered at p, such that the Riemannian metric $g_{ij}(x)$ calculated at p coincides with δ_{ij}. Let $U(\delta)$ be the open (convex) ball centered at p and having the radius δ:

$$U(\delta) = \{x \in M : \rho(p,x) < \delta\}. \tag{8.5}$$

Any two points x and y from the ball are joined by a unique geodesic γ lying in $U(\delta)$. At that time, $\rho(x,y) = \mathrm{length}(\gamma)$. Thus, the ball $U(\delta)$ is a metric space with intrinsic metric; that is, the distance between the points equals the greatest lower bound of the curves‘ lengths over all the measurable curves joining the points. Notice that in terms of the coordinates (x^i) the ball $U(\delta)$ is defined as follows:

$$U(\delta) = \{(x^1)^2 + \cdots + (x^d)^2 < \delta^2\}. \tag{8.6}$$

Therefore, if we define the Euclidean distance ρ_e in $U(\delta)$ (in terms of the normal coordinates (x^i)), then the metric space $(U(\delta), \rho_e)$ also is the space with intrinsic metric generated by the Euclidean metric δ_{ij}.
Since the Riemannian metric $g_{ij}(x)$ depends smoothly on $x \in U(\epsilon)$, then for any ϵ, $1/d^2 > \epsilon > 0$, there exists a $\delta > 0$ such that

$$|g_{ij}(x) - \delta_{ij}| < \epsilon \tag{8.7}$$

for all points $x \in U(\delta)$. The latter implies the following proposition which gives an important interrelation between Riemannian and Euclidean metrics.

Lemma 8.2.1 *Let $\|v\|_g$ be the length of the tangent vector $v \in T_xM$ with respect to the Riemannian metric g_{ij}, and let $||v||_e$ be the length of v with respect to the Euclidean metric δ_{ij}. If for any i and j the inequality (8.7) holds, then*

$$\sqrt{1-d^2\epsilon} \cdot ||v||_e \leq ||v||_g \leq \sqrt{1+d^2\epsilon} \cdot ||v||_e. \tag{8.8}$$

Using the definition of the distance between a pair of points of a connected Riemannian manifold, we obtain the following result.

Lemma 8.2.2 *Let M be an arbitrary connected d-dimensional Riemannian manifold, and let $U(\delta)$, ρ, and ρ_e be as above. Then for an arbitrary ϵ, $1/d^2 > \epsilon > 0$, there exists a $\delta > 0$ such that for all points $x, y \in U(\delta)$*

$$\sqrt{1-d^2\epsilon} \cdot \rho_e(x,y) \leq \rho(x,y) \leq \sqrt{1+d^2\epsilon} \cdot \rho_e(x,y). \tag{8.9}$$

8.3 Riemannian Manifolds

Since the Steiner ratio is evidently the same for any convex open subsets of $I\!R^d$, 8.2.2 and 6.2.7 lead to the following result.

Corollary 8.3.1 *Let M be an arbitrary d-dimensional Riemannian manifold, let $U(\epsilon) \subseteq M$ be an open ball of a small radius ϵ. By ρ we denote the metric on M generated by the Riemannian metric. Then*

$$\sqrt{\frac{1-d^2\epsilon}{1+d^2\epsilon}} \cdot m(I\!R^d) \leq m(U(\epsilon),\rho) \leq \sqrt{\frac{1+d^2\epsilon}{1-d^2\epsilon}} \cdot m(I\!R^d), \tag{8.10}$$

where $m(I\!R^d)$ stands for the Steiner ratio of the d-dimensional Euclidean space $I\!R^d$.

This fact will be helpful to prove the following theorem.

Theorem 8.3.2 *(Ivanov, Tuzhilin, Cieslik [211]) The Steiner ratio of an arbitrary d-dimensional connected Riemannian manifold M does not exceed the Steiner ratio of $I\!R^d$.*

Sketch of the *proof.* For some decreasing sequence $\{\epsilon_i\}$ of positive real numbers with $\epsilon_i < \epsilon$ for any index i, where $\epsilon_i \to 0$, we consider a family of nested subsets $X_i = U(\epsilon_i)$.
In view of the convexity of unit balls $(U(\epsilon), \rho_\epsilon)$ we have $m(U(\epsilon), \rho_\epsilon) = m(I\!R^d)$. The unit ball $U(\epsilon)$ with the intrinsic metric ρ' is a subspace of (M, ρ). 8.3.1 implies that

$$m(X_i,\rho) \leq \sqrt{\frac{1+d^2\epsilon}{1-d^2\epsilon}} \cdot m(I\!R^d). \tag{8.11}$$

Since $\sqrt{\frac{1+d^2\epsilon}{1-d^2\epsilon}} \to 1$ as $i \to \infty$ due to the choice of $\{\epsilon_i\}$, we get

$$\inf_i m(X_i,\rho) \leq m(I\!R^d). \tag{8.12}$$

But, due to 6.3.1 we have:

$$m(M, \rho) \leq \inf_i m(X_i, \rho). \tag{8.13}$$

□

Applying proposition 6.3.4 gives

Theorem 8.3.3 *(Ivanov, Tuzhilin, Cieslik [211]) Let $\pi : W \to M$ be a locally isometric covering of connected Riemannian manifolds. Then the Steiner ratio of the base M of the covering is more than or equal to the Steiner ratio of the total space W.*

Corollary 8.3.4 *Assume that 4.6.1 is true. Then the Steiner ratio for all closed two-dimensional surfaces is equal to $\sqrt{3}/2$.*

And

Corollary 8.3.5 *The Steiner ratio for a) a flat two-dimensional torus; b) a flat Klein bottle; and c) a projective plane having constant positive curvature is equal to then Steiner ratio of the Euclidean plane.*

Idea of the *proof.* It follows from theorems 8.3.2 and 8.3.3 that the Steiner ratio equals that of the Euclidean plane.
From Du and Hwang's theorem [129] and [125] saying that the Steiner ratio of the Euclidean plane equals $\sqrt{3}/2$; and also from Rubinstein and Weng's theorem [312] saying this for the two-dimensional sphere with constant positive curvature.

□

Thus, taking the results of Rubinstein and Weng [312] into account, the Steiner ratio is computed now for all closed surfaces having nonnegative curvature.

8.4 Lobachevsky Spaces

Let us consider the Poincaré model of the Lobachevsky plane $L^2(-1)$ with constant curvature -1. We recall that this model is a flat disk of radius 1 centered at the origin of the Euclidean plane with Cartesian coordinates (x, y), and the metric ds^2 in the disk is defined as follows:

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \tag{8.14}$$

It is well-known that for each regular triangle in the Lobachevsky plane the circumscribed circle exists. The radii emitted out of the center of the circle to the vertices of the triangle form the angles of 120^o.

Let r be the radius of the circumscribed circle. The cosine rule implies that the length a of the side of the regular triangle can be calculated as follows:

$$\cosh a = \cosh^2 r - \sinh^2 r \cos\frac{2\pi}{3} = 1 + \frac{3}{2}\sinh^2 r.$$

It is easy to verify that for such triangle the length of an MST equals $2a$ and the length of an SMT equals $3r$. Therefore, the Steiner ratio $m(r)$ for the regular triangle inscribed into the circle of radius r in the Lobachevsky plane $L^2(-1)$ has the form

$$m(r) = \frac{3}{2}\cdot\frac{r}{\operatorname{arccosh}(1+\frac{3}{2}\sinh^2(r))}.$$

It is easy to calculate that the limit of the function $m(r)$ as $r \to \infty$ is equal to $3/4$. Consequently,

Theorem 8.4.1 *(Ivanov, Tuzhilin, Cieslik [212]) The Steiner ratio of the Lobachevsky space with curvature -1 does not exceed $3/4$.*

And,

Theorem 8.4.2 *(Ivanov, Tuzhilin, Cieslik [212]) The Steiner ratio of an arbitrary surface of constant negative curvature -1 is strictly less than $\sqrt{3}/2$.*

Proof. It is easy to see that the Taylor series of the function $m(r)$ at $r = 0$ has the following form:

$$\frac{\sqrt{3}}{2} - \frac{r^2}{16\sqrt{3}} + O(r^4).$$

Therefore, $m(r)$ is strictly less than $\sqrt{3}/2$ in some interval $(0, \epsilon)$. The latter means that for sufficiently small regular triangles on the surfaces of constant curvature -1, the relation of the lengths of an SMT and an MST is strictly less than $\sqrt{3}/2$.

□

These results have been enforced for specific spaces.

Theorem 8.4.3 *(Innami, Kim [203]) The Steiner ratio of a simply connected manifold of negative constant curvature without boundary equals $1/2$.*

Idea of the *proof.* We consider the Poincaré disk $H = \{(x,y) : x^2 + y^2 < 1\}$ with the Riemannian metric

$$ds^2 = 4\frac{dx^2 + dy^2}{c(1 - x^2 - y^2)^2}, \tag{8.15}$$

for a positive c. Any complete simply connected manifold of negative constant curvature $-c$ without boundary is isometric to H.

Let n be an integer greater than 2. Let O be the origin in H and $\gamma_i : [0, \infty) \to H$ geodesic rays for $i = 1, \ldots, n$ such that

$$\begin{aligned} \gamma_i(0) &= O, \\ \text{angle of } (\gamma_i'(0), \gamma_{i+1}'(0)) &= \frac{2\pi}{n}, \quad \text{and} \\ \gamma_{n+1} &= \gamma_1. \end{aligned}$$

Let $N(s) = \{\gamma_i(s) : i = 1, \ldots, n\}$ for a positive s.
$T(\gamma_i(s), \gamma_{i+1}(s))$ denotes the minimal subtree from $\gamma_i(s)$ to $\gamma_{i+1}(s)$ in the SMT of $N(s)$. Then it holds

$$\lim_{s \to \infty} \frac{L(T(\gamma_i(s), \gamma_{i+1}(s)))}{\rho(\gamma_i(s), \gamma_{i+1}(s))} = 1. \tag{8.16}$$

With the choice of $N(s)$ we have

$$L(\text{MST for } (N(s))) = (n-1)\rho(\gamma_1(s), \gamma_2(s)). \tag{8.17}$$

Consequently,

$$\begin{aligned} \frac{L(\text{ SMT for } (N(s)))}{L(\text{ MST for } (N(s)))} &= \frac{1}{2} \cdot \frac{\sum_{i=1}^n L(T(\gamma_i(s), \gamma_{i+1}(s))}{(n-1) \cdot \rho(\gamma_1(s), \gamma_2(s))} \\ &= \frac{1}{2} \cdot \frac{n}{n-1} \cdot \frac{\sum_{i=1}^n L(T(\gamma_i(s), \gamma_{i+1}(s))}{n \cdot \rho(\gamma_1(s), \gamma_2(s))} \\ &= \frac{1}{2} \cdot \frac{n}{n-1} \cdot \frac{\sum_{i=1}^n L(T(\gamma_i(s), \gamma_{i+1}(s))}{\sum_{i=1}^n \rho(\gamma_i(s), \gamma_{i+1}(s))}. \end{aligned}$$

Then it follows by (8.16):

$$\lim_{s \to \infty} \frac{L(\text{SMT for } (N(s)))}{L(\text{MST for } (N(s)))} = \frac{n}{2(n-1)}. \tag{8.18}$$

Since this must be true for all integers $n > 2$, we get the assertion.

$\square$

It seems that the Steiner ratio of Riemannian manifolds equals the Steiner ratio of the Euclidean spaces and of Lobachevsky spaces equals 1/2. Is this observation really true?

Chapter 9

Related Questions

The general theme of the present script is the problem of finding cheapest networks linking a set of points. In the center there is Steiner's Problem, which is not only a single question; there are several modifications of the problem. There are an almost unlimited number of such relatives, [282].

9.1 k-SMT's

We investigate the problem of finding a k-SMT, which allows at most k Steiner points in the shortest tree. This problem was introduced independently by Cieslik [68] in 1982 and Georgakopoulos and Papadimitriou [166] in 1987.[1] In general, the combinatorial structures of k-SMT's and SMT's are quite different. In particular, in k-SMT's we find Steiner points of degree higher than in SMT's. To make essential results we must restrict the class of metric spaces under consideration.

Assumption: There is a positive integer $c = c(X, \rho)$, depending on the space only, such that the degree of any Steiner point in each k-SMT for a given set in (X, ρ) is at most c.

Note that, a) The number $c = c(X, \rho)$ can be determined for a 1-SMT; and b) If $m(X, \rho) = 1$, then any SMT and any k-SMT is an MST. Otherwise, if $m(X, \rho)$ is less than one, then $c(X, \rho) \geq 3$.
For the number c for some metric spaces see [83].[2]

[1] While a k-SMT can be used as a heuristic, the time required for construction is a large polynomial of the number n of points unless k is very small, but then the performance cannot be good. Thus the k-SMT is more a generalization of the SMT rather than a heuristic.

[2] Particularly, for Banach-Minkowski spaces $M_d(B)$ such a value always exists. This is shown in [10] and [72], and a complete discussion is given by [344]. For $d = 2$ we have $c \leq 5$ except the plane which is normed by an affinely regular hexagon, where $c = 6$, [340]. It seems, that the classification of all planes in which the maximal degree is exactly 5 is too hard, because we even have difficulties to decide this question in the Euclidean plane [71], where $c = 4$, see [311]. We have $c = 6$ possible for a plane normed by an affinely regular hexagon, $c = 5$ for the plane with rectilinear distance and

For a fixed integer k, a k-SMT for a finite finite set of points in a metric space which satisfies the assumption can be found in polynomially bounded time, [83].

Let k and k' be integers with $0 \leq k' \leq k \leq \infty$. We define the restricted Steiner ratio of the metric space (X, ρ) by

$$m(X,\rho)(k:k') = \inf\left\{\frac{L(k\text{-SMT for } N)}{L(k'\text{-SMT for } N)} : N \text{ is a finite set in } (X,\rho)\right\}. \tag{9.1}$$

(For $k < k'$ this quantity is undefined.) Since $m(X,\rho)(\infty, 0) = m(X,\rho)$ it holds

$$1 \geq m(X,\rho)(k:k') \geq m(X,\rho) \geq 1/2. \tag{9.2}$$

The ratio $m(X,\rho)(k:k-1)$ is of special interest. To estimate it, we remember the local version of Steiner's Problem, the so-called Fermat's Problem: Let N be a finite set of points in (X,ρ). Determine a point w in the space such that the function

$$F_N(w) = \sum_{v \in N} \rho(v, w) \tag{9.3}$$

is minimal. Each point which minimizes the function F_N is called a Torricelli point for N in (X,ρ).

Lemma 9.1.1 *Let N be a finite set of n points in a metric space. Let q be a Torricelli point and T_o be an MST for N. Then*

$$\frac{F_N(q)}{L(T_o)} \geq \frac{n}{2n-2}.$$

Proof. Let $N = \{v_1, \ldots, v_n\}$.
If q is in N, then $F_N(q) \geq L(T_o)$, and the ratio is at least one.
Now, we assume that q is not in N. Without loss of generality, $\rho(v_1, v_n)$ is the greatest distance between points of N. Hence,

$$\begin{aligned}
2(n-1)F_N(q) &= (n-1)\left(\sum_{i=1}^{n}\rho(v_i,q) + \sum_{j=1}^{n}\rho(v_j,q)\right)\\
&\geq (n-1)\left(\sum_{i=1}^{n-1}\rho(v_i,v_{i+1}) + \rho(v_1,v_n)\right)\\
&\geq (n-1)L(T_o) + (n-1)\rho(v_1,v_n)\\
&\geq (n-1)L(T_o) + \sum_{i=1}^{n-1}\rho(v_i,v_{i+1})\\
&\geq (n-1)L(T_o) + L(T_o)\\
&= nL(T_o).
\end{aligned}$$

$c = 4$ for the Euclidean plane

□

Theorem 9.1.2 *(Cieslik [87]) In a metric space* (X,ρ) *which satisfies the assumption, it holds*

$$m(X,\rho)(k:k-1) \geq \frac{k}{k+2-\frac{4}{c(X,\rho)}} \tag{9.4}$$

for all integers $k > 0$.

Proof. Let $T = (V,E)$ be a k-SMT for N. Then the degree of any Steiner point v is at most $c = c(X,\rho)$.
If $|V| < |N| + k$, then T also is a $(k-1)$-SMT, and the ratio equals one. Now, we assume that $|V| = |N| + k$.
Let $q \in V \setminus N$ such that the star T_s induced by q and its set V_s of neighbors in T has minimal length. Let T_c be an MST for V_s. Clearly, $L(T_s) \leq L(T_c)$. On the other hand, by 9.1.1 and the fact that the real function $x/(2x-2)$ is monotonically decreasing it follows

$$L(T_s) \geq \frac{c}{2c-2} \cdot L(T_c).$$

T' is the tree built up by T with T_c instead of T_s. Then T' is a tree with at most $k-1$ Steiner points. On the one hand,

$$\begin{aligned} L((k-1)\text{-SMT for } N) &\leq L(T') \\ &= L(T) - L(T_s) + L(T_c) \\ &\leq L(k\text{-SMT for } N) - L(T_s) + (2-2/c)L(T_s) \\ &= L(k\text{-SMT for } N) + (1-2/c)L(T_s). \end{aligned}$$

On the other hand,

$$\begin{aligned} L(k\text{-SMT for } N) &= L(T) \\ &\geq \frac{1}{2} \cdot \sum_{v \in V \setminus N} L(\text{star induced by } v \text{ and its neighbors}) \\ &\geq \frac{1}{2} \cdot \sum_{v \in V \setminus N} L(T_s) \\ &= \frac{k \cdot L(T_s)}{2}. \end{aligned}$$

These two inequalities imply the assertion.

□

The theorem shows that the best addition of k Steiner points to the initial set of given points cannot improve the approximation drastically in comparison to the best addition of $k-1$ Steiner points if k is a large number. In particular, combining (9.2), 9.1.2 and 3.2.1, this is satisfied if

$$k \geq \frac{m(X,\rho)}{1-m(X,\rho)} \cdot \left(2 - \frac{4}{c(X,\rho)}\right) > \frac{1}{3(1-m(X,\rho))}.$$

In other terms: The "relative defect" going from a $(k-1)$-SMT to a k-SMT for a finite set in a metric space tends to zero, when k runs to infinity.
For instance, we consider the d-dimensional affine space with rectilinear distance. Let $N = \{\pm(1,0,\ldots,0),\ldots,\pm(0,\ldots,0,1)\}$. Clearly, an MST T for N has length $4d-2$, and the origin is a Torricelli point for N. This implies $F_N/L(T) = d/(2d-1)$. In other words,

$$m(k:k-1) \geq \frac{k}{k+2-\frac{2}{d}} \tag{9.5}$$

for $k \geq 1$. Hence, the inequality in 9.1.2 is the best possible one in the class of all metric spaces.
Kallmann [220] proves $m(1:0) = \sqrt{3}/2$ for the Euclidean plane, that means in the seond inequality in (9.2) equality holds.

9.2 The Star Ratio

In the sense of the former section it is likely to consider the problem of the so-called star ratio, [140]: Let N be a set of n points in a metric space (X,ρ). A Steiner star for N connects an arbitrary median point to all points of N, while a star connects a point v to the remaining $n-1$ points in N. To find a Steiner star with minimum length is the same as to solve Fermat's problem; the median point is the Torricelli point. A star is a specific tree. Then the star ratio is defined as

$$r(X,\rho) = \inf\left\{\frac{L(\text{Steiner star for } N)}{L(\text{star for } N)} : N \subseteq X, |N| < \infty\right\}. \tag{9.6}$$

In the d-dimensional Euclidean space, we consider the set N of $d+1$ nodes of a regular simplex with edges of unit length. Then a star for N has the length d. In view of 4.15.1 the sphere that circumscribes N has the radius $R(N) = \sqrt{d/(2d+2)}$. With the center of this sphere as Torricelli point, we find a Steiner star T interconnecting N with the length $L(B(2))(T) = (d+1)R(N)$. Altogether

Theorem 9.2.1 *For the d-dimensional Euclidean space it holds*

$$r(M_d(B(2))) \leq \sqrt{\frac{1}{2}+\frac{1}{2d}}. \tag{9.7}$$

On the other hand, with help of a deep investigation, there is a tight bound:

Theorem 9.2.2 *(Fekete, Meijer [145]) For Euclidean spaces it holds*

$$r(M_d(B(2))) \geq \frac{1}{\sqrt{2}} = 0.70710\ldots. \tag{9.8}$$

To prove 9.2.1 we used the same argument as in 4.12.1, to estimate the Steiner ratio of Euclidean spaces, but there we find not sharp bounds. This is not a surprise, since a Steiner star cannot be a SMT for the nodes of a simplex, because an Euclidean

SMT has only Steiner points of degree three, see 4.2.1. On the other hand, an MST is not necessarily a star.

Since a Steiner star is a specific tree interconnecting a set N of points, we have L(Steiner star for N) $\geq L$(SMT for N), and therefore

Observation 9.2.3

$$\frac{L(X,\rho)(\textit{Steiner star for } N)}{L(X,\rho)(\textit{MST for } N)} \geq r(X,\rho). \tag{9.9}$$

Brenner, Vygen [47] discussed this question for the rectilinear plane. In particular,

Example 9.2.4 *For a set N with n given points in the rectilinear plane*

$$\frac{L(\textit{Steiner star for } N)}{L(\textit{MST for } N)} \geq \begin{cases} \frac{2}{3} & : \textit{ in general} \\ \frac{5}{6} & : \ n=5 \\ 1 & : \ n \geq 6 \end{cases}$$

This behavior is a consequence of bounded vertex-degrees in an MST.[3]

9.3 SMT(α)

We saw: Let N be a finite set of points in a metric space. Then the relative defect when going from a $(k-1)$-SMT to a k-SMT for N tends to zero if k runs to infinity.

[3]More generally, let B be a unit ball of the d-dimensional affine space A_d. A translation of B is a congruent copy of B moved to another location in space while the original orientation of B is preserved. The Hadwiger number (or kissing number) $H_d(B)$ of the Banach-Minkowski space $M_d(B)$ is the maximum number of nonoverlapping translations of B which can be brought into contact with B. Grünbaum [176] showed that

$$H_d(B) = \max\{|W| : W \subseteq \text{boundary of } B, ||w-w'||_B \geq 1, w, w' \in W, w \neq w'\}. \tag{9.10}$$

Theorem 9.2.5 *(Cieslik [83], [84]) For a Banach-Minkowski space the Hadwiger number is a sharp upper bound for the vertex degrees in any MST (or 1-SMT).*

This fact was independently proved by Robins and Salowe [305] for $\mathcal{L}_p^d$.
Unfortunately, our knowledge about the exact values for the Hadwiger number is limited:

(a) (Hadwiger [181]) $H_d(B) \leq 3^d - 1$.

(b) (Groemer [172]) $H_d(B) = 3^d - 1$ if and only if B is a parallelepiped.

Odlyzko and Sloane [285] find good estimates of $H_d(B(2))$ for the dimension d between 1 and 24. And Larman, Zong [240] show that the Hadwiger number $H_d(B(p))$ grows exponentially in d:

$$3^{0.1072\ldots d(1+o(1))} \leq H_d(B(p)) \leq 3^d - 1. \tag{9.11}$$

Moreover, the Hadwiger numbers for planar convex bodies are completely determined:

Observation 9.2.6 *(Grünbaum [176])*

$$H_2(B) = \begin{cases} 8 & : \ B \textit{ is a parallelogram} \\ 6 & : \ \textit{otherwise.} \end{cases}$$

Now, we will use this fact to estimate the number k for k-SMT's depending on the number α for an SMT(α). That means we consider the following problem: Let N be a finite set of points in a metric space (X, ρ) and α a nonnegative number. Look for a network $T = (V, E)$ interconnecting N such that

$$\mathcal{C}(T) = \alpha \cdot |V \setminus N| + L(T) \tag{9.12}$$

is minimal. $\mathcal{C}(T)$ is called the cost of the tree T.[4]
Consider the following example: The four points $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$, which are the corners of a square in the Euclidean plane:

shortest tree	length $L(\cdot)$	number of Steiner points
MST T_0	$3 \cdot \sqrt{2} = 4.242...$	0
1-SMT T_1	4	1
SMT T_2	$\sqrt{6} + \sqrt{2} = 3.863...$	2

Then T_1 is an SMT(0.2) and T_0 is an SMT(0.4).[5]

Denote by T_k a k-SMT for N. Then

$$\mathcal{C}(T_k) \leq \alpha \cdot k + L(T_k). \tag{9.13}$$

If T_k contains at most $k-1$ Steiner points, then we know that it is also a $(k-1)$-SMT for N, and it holds

$$L(T_k) = L(T_{k-1}). \tag{9.14}$$

In any case, we have 9.1.2 in

$$L(T_{k-1}) \geq L(T_k) \geq \frac{k}{k+\delta} \cdot L(T_{k-1}), \tag{9.15}$$

whereby the quantity

$$\delta = \delta(X, \rho) = 2 - \frac{4}{c(X, \rho)} \tag{9.16}$$

is a positive real number: $\delta \geq \frac{2}{3}$, since we have $c \geq 3$, but less than 2.

Now, we consider the costs of the k-SMT's. If $T_{k-1} = T_k$, which means that a k-SMT uses at most $k-1$ Steiner points, then $\mathcal{C}(T_{k-1}) = \mathcal{C}(T_k)$. In the other case we find

$$\mathcal{C}(T_k) = \alpha \cdot k + L(T_k). \tag{9.17}$$

[4]The more general question for a given finite set N of points in (X, ρ), and for nonnegative real numbers α_1, α_2, to find a connected graph $G = (V, E)$ such that $N \subseteq V$ and the quantity

$$\mathcal{C}(G) = \mathcal{C}(\alpha_1, \alpha_2)(G) = \alpha_1 \cdot |V \setminus N| + \alpha_2 \cdot L(G)$$

is minimal; can be reduced to an SMT(α).

[5]Underwood [358] presents many properties of SMT(α)'s in the Euclidean plane and consequently a modified Melzak procedure which computes an SMT(α) for a given set of points.

We are interested in the condition $\mathcal{C}(T_k) \le \mathcal{C}(T_{k-1})$. Recalling (9.13) and (9.17), we see that this condition is equivalent to $\alpha \cdot k + L(T_k) \le \alpha \cdot (k-1) + L(T_{k-1})$. Hence, the insertion of a new Steiner point is only sensible if the difference between the lengths of the trees is at least the value of the parameter α:

$$\alpha \le L(T_{k-1}) - L(T_k). \tag{9.18}$$

Furthermore, in view of (9.15), we have

$$L(T_{k-1}) - L(T_k) \le \frac{\delta}{k} L(T_k) \tag{9.19}$$

Both inequalities (9.18) and (9.19) imply

$$\alpha \le \frac{\delta}{k} L(T_k). \tag{9.20}$$

In other terms, the insertion of a new Steiner point is only sensible if (9.20) holds.

Theorem 9.3.1 *If we are looking for an SMT(α) for a set N of given points in a metric space (X, ρ), which satisfies the assumption of the upper bound, then we are only interested in the k-SMT's for N with*

$$k \le \frac{\delta}{\alpha} \cdot L(MST\ for\ N),$$

where δ is defined by (9.16).

9.4 A monotone iterative Procedure to find shortest Trees

An MST is not the shortest possible interconnecting network if new points may be added. So we introduce the following Solomon-like compromise between what is desirable and what is practical to compute.

I. A natural idea is to use the ideas of the former sections as an approximation-strategy for Steiner's Problem, [60], [93], [219], [331], and [382]. In any case we apply a procedure to find a 1-SMT repeatedly.[6] This means: Start with the given finite set, and successively add Steiner points, one Steiner point at a time. Note that once added, a Steiner point cannot be removed. We call such a method a monotonic iterative algorithm. During the course of such an algorithm, a sequence

$$N = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$$

of sets of points is constructed such that for all integers $k \ge 0$

$$L(X, \rho)(\text{MST for } V_{k+1}) \le L(X, \rho)(\text{MST for } V_k).$$

[6] For techniques to solve the 1-SMT-Problem in some specific spaces compare [75], [76], [166] and [219].

It is, however, possible that such constructions do not produce an SMT: Salowe and Warme [316] gave an example in the plane with rectilinear distance.
With this in mind we find an iterative approximation of Steiner's Problem:

Given: a) The following quantities for the metric space (X, ρ): the Steiner ratio $m(X, \rho)$, and the degree bound $c(X, \rho)$.
b) A method to find a Torricelli point for any set of at most $c(X, \rho)$ points.

Input: A finite set N of points in X.

Choose: Two performance error bounds ϵ_a and ϵ_r (for absolutely, *a priori*, and relatively, *a posteriori*, performance ratios, respectively).

Algorithm: Do:

1. $k := 0$;
determine an MST $T^{(0)} = (V_0, E_0)$ for N;
$L_{(0)} := L(X, \rho)(T^{(0)})$;
2. repeat
$k := k + 1$;
determine a 1-SMT $T^{(k)} = (V_k, E_k)$ for V_{k-1};
$L_{(k)} := L(X, \rho)(T^{(k)})$;
until
$$m(X, \rho) \cdot L_{(0)} \leq \epsilon_a$$
or
$$\frac{L_{(k-1)} - L_{(k)}}{L_{(k-1)}} \leq \epsilon_r.$$
(Particularly, this is valid if $n - 2$ iterations have been executed.)

II. A very similar approach in Banach-Minkowski planes is the following : After creating a triangulation for the set N, find an MST and improve it by adding some Steiner points.[7] Originated in [331] for the Euclidean plane, we generalize this idea.

Algorithm 9.4.1 *Let N be a finite set of at least three points in the plane with p-norm. Then*

1. *Construct the triangulation $G = (N, E)$ for N;*
2. *Determine an MST $T = (N, E')$ for N with $E' \subseteq E$;*
$V' := N$;

[7] A triangulation for N is a partition of the convex hull of N into triangles, such that the vertices of these triangles are exactly the points of N. Since a triangulation is a connected and planar graph it contains a spanning tree and has at most $3n - 6 = O(n)$, $n = |N|$, edges. In such a "sparse" network it is easy to find an MST. More facts about this topic are in [36].

3. *If a triangle* $\{v_1, v_2, v_3\}$ *of* G *has the property that two of its sides are in* E' *(without loss of generality* $\underline{v_1v_3}, \underline{v_2v_3} \in E \cap E'$*), then find a Torricelli point* q*;*
If $\rho(v_1, q) + \rho(v_2, q) + \rho(v_3, q) < \rho(v_1, v_3) + \rho(v_2, v_3)$ *then do*
$V' := V' \cup \{q\}$*;*
$E' := E' \cup \{\underline{v_iq} : i = 1, 2, 3\} \setminus \{\underline{v_1v_3}, \underline{v_2v_3}\}$*;*
$T := (V', E')$*.*

In the algorithm we adopted the idea of a greedy improvement. It starts with a triangulation instead of an MST. Assuming that desired chosen triangulations contains an MST, the algorithm creates a shorter tree.

Theorem 9.4.2 *Let* N *be a finite set of points and let* T' *be a tree determined by the algorithm 9.4.1. Then in the plane with p-norm*

$$1 \geq \frac{L(T')}{L(MST\ for\ N)} \geq m^3(2, p).$$

Proof.

$$L(T') = \sum_{\underline{vv'} \in E'} \rho(v, v') = \sum_{q \in V' \setminus N} F_{N(q)}(q) + L(G')$$

where $N(q)$ is the set of all neighbors of q in T' and $G' = (N, \{\underline{vv'} : v, v' \in N\})$ is a forest for N.
Let T_q be an MST for $N(q)$, and we continue:

$$\begin{aligned} L(T') &\geq \sum_{q \in V' \setminus N} m^3(2, p) \cdot L(T_q) + L(G') \\ &= m^3(2, p) \sum_{q \in V' \setminus N} L(T_q) + L(G') \\ &\geq m^3(2, p) \left(\sum_{q \in V' \setminus N} L(T_q) + L(G') \right). \end{aligned}$$

Since $G' \cup \bigcup_{q \in V' \setminus N} T_q$ is a tree interconnecting N without Steiner points, the assertion follows.

□

9.4.1 can be fails by two reasons: On one hand we know, in view of 3.2.4, the ratio is at least 3/4. But, by 4.7.7, there are values for p such that the Steiner ratio is less than 3/4. On the other hand, the algorithm produces only Steiner points of degree three.[8]

[8] Foulds [153] gives a similar approach for constructing trees in the space of sequences with Hamming distance.
A related method, for creating the so-called phylogenetic median-joining network, is given by Bandelt et al. [20]. The algorithm begins with a minimum spanning network. Aiming at parsimony, subsequently added few consensus sequences (as a kind of Steiner points) of three mutually close sequences. This procedure will repeated.

9.5 Greedy Trees

Since Steiner's Problem has been shown to be $\mathcal{NP}$-hard in most the of metric spaces, we are interested in approximations of SMT's by efficiently computable trees. The performance ratio of any approximation $\mathcal{M}$ in metric spaces is defined by

$$m_{\mathcal{M}}(X,\rho) = \inf\left\{\frac{L(\text{SMT for } N)}{L(\mathcal{M}(N))} : N \text{ a finite set in } X \text{ for which an SMT exists}\right\}. \tag{9.21}$$

Then,

$$m(X,\rho) = m_{\text{MST}}(X,\rho). \tag{9.22}$$

Smith and Shor [333] introduced the notion of a so-called Greedy Tree (GT) for a set N of points in a Euclidean space as follows:

1. Start with all points of N, regarded as a forest of $n = |N|$ single vertices;
2. At any stage, add the shortest possible segment to the current forest, which causes two trees to merge;
3. Continue until the forest is completely merged into one tree.

Greedy Trees are simple to construct and a GT $T = (V, E)$ for N in a Euclidean space is an MST for V. Moreover, T is no longer than an MST for N. Hence,

$$\frac{L(\text{SMT for } N)}{L(T)} \geq \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} \geq m(X,\rho),$$

Smith, Shor [333] conjectured that the ratio between an SMT and a GT is greater than the Steiner ratio of the Euclidean plane:

$$\inf\left\{\frac{L(\text{SMT for } N)}{L(\text{GT for } N)} : N \subseteq \mathcal{L}_2^2 \text{ is a finite set}\right\} = \frac{2\sqrt{3}}{2+\sqrt{3}} = 0.9282\ldots.$$

Du [134], however, disproves this conjecture in the following way: He shows that we can always choose a finite set N of points such that the ratio between the length of an SMT for N and the length of a greedy tree for the same set of points can be arbitrarily close to the Steiner Ratio $\sqrt{3}/2$. He proposes the new conjecture that the better performance 0.9282... is achieved by the greedy trees over all permutations of the points as choice of the construction.

Even in this form, however, it may be useful for problems such as connecting new points to an existing network, [354].

In high dimensions the advantage of GT's over MST's can become quite pronounced: Let $N(d)$ be the nodes of a regular simplex of unit side length in the d-dimensional Euclidean space. Then an MST has length d, and a GT for $N(d)$ has length

$$\sum_{1\leq k\leq d} \sqrt{\frac{k+1}{2k}} \approx 0.7071 \cdot d \tag{9.23}$$

for $d \to \infty$. On the other hand, we have an upper bound for the Steiner ratio of $0.66984 \cdot d$, see 4.12.2.

9.6 Component-size bounded Steiner Trees

All our general approaches to approximate Steiner's Problem in arbitrary metric spaces, including finite ones, have error 2. Is there a significantly better method? In 1992 Zelikovsky [383] made the first breakthrough by proposing an approximation method for Steiner's Problem in graphs. He uses trees that can contain Steiner points, but not in an arbitrary sense: Let N be a finite set of points in a metric space (X, ρ). Let $T = (V, E)$ be a tree interconnecting N. For such trees we assume that the degree of each given point is at least one and the degree of each Steiner point in $V \setminus N$ is at least three. However, a given point v in such a tree may not be a leaf. Then T can be decomposed (by splitting at the given point) into several smaller trees, so that given points only occur as leaves:

1. Define $G = (V \setminus \{v\}, E \setminus \{\underline{vv'} : v' \text{ is a neighbor of } v\})$.
   (G is a forest with $g(v)$ components $G_i = (V_i, E_i)$, $i = 1, \ldots, g(v)$.)
2. Define for $i = 1, \ldots, g(v)$ the graph
   $G_{(i)} = (V_i \cup \{v_i\}, E_i \cup \{\underline{v_i v'} : v' \text{ is a neighbor of } v \text{ in } G \text{ and } v' \text{ is in } V_i\})$,
   where v_i is not in V.

In this way, every tree interconnecting N is decomposed into so-called full components. The size of a full component is the number of given points in the full component.

A k-size tree for N is a tree interconnecting all points of N with all full components of size of at most k. A k-size SMT is the shortest one among all k-size trees. For $k = 2$ we look for an MST. For every $k \geq 4$ this problem is $\mathcal{NP}$-hard, [301].

Clearly, we are interested in the greatest lower bound for the ratio between the lengths of an SMT and a k-size SMT for the same set of points in a metric space:

$$m^{(k)} = m^{(k)}(X, \rho) = \inf \left\{ \frac{L(\text{SMT for } N)}{L(k\text{-size SMT for } N)} : N \subseteq (X, \rho) \text{ is a finite set} \right\}. \tag{9.24}$$

This quantity is called the k-size-Steiner ratio of the metric space (X, ρ).
In any metric space (X, ρ) a 2-size SMT is an MST. Hence,

$$m^{(2)}(X, \rho) = m(X, \rho). \tag{9.25}$$

Furthermore,

Theorem 9.6.1 *For the k-size-Steiner ratio $m^{(k)}$, $k > 2$, the following is known:*

a) *(Zelikovsky [383]) For any metric space (X, ρ) it holds that*

$$m^{(3)}(X, \rho) \geq \frac{3}{5} = 0.6. \tag{9.26}$$

(Du [133]) This lower bound is the best possible one over the class of all metric spaces.

b) *(Du [126], Du, Zhang [128]) For any metric space (X,ρ) it holds that*

$$m^{(k)}(X,\rho) \geq \frac{r}{r+1}, \tag{9.27}$$

where $r = \lfloor \log_2 k \rfloor$.

Zelikovsky [383], [384] shows that there exists a polynomial-time approximation $\mathcal{M}$ for Steiner's Problem in a metric space (X,ρ) with performance ratio

$$\text{error}(\mathcal{M}) = \frac{1}{2} \cdot \left\{ \frac{1}{m^{(3)}(X,\rho)} + \frac{1}{m^{(2)}(X,\rho)} \right\}, \tag{9.28}$$

provided that an SMT for three given points can be computed in polynomial time.[9] In view of 9.6.1(a) and 3.2.1, we obtain

Corollary 9.6.2 *There is an approximation algorithm for Steiner's problem with approximation error $\frac{11}{6}$.*

Using a similar idea, Berman and Ramaiyer [29] presents a polynomial-time approximation $\mathcal{M}_k$ with performance ratio

$$\text{error}(\mathcal{M}_k) \geq \frac{1}{1\cdot 2} \cdot \frac{1}{m^{(2)}(X,\rho)} + \frac{2}{2\cdot 3} \cdot \frac{1}{m^{(3)}(X,\rho)} + \frac{1}{3\cdot 4} \cdot \frac{1}{m^{(4)}(X,\rho)} + \ldots, \tag{9.29}$$

provided that for any k an SMT for k points can be computed in polynomial time. Clearly, we are interested in the k-size-Steiner ratio for specific spaces. For the plane with rectilinear distance we have

k	$m^{(k)} =$	Source
$= 2$	$2/3$	Hwang, [199]
$= 3$	$4/5$	Berman and Ramaiyer, [29]
≤ 4	$2k - 1/2k$	Borchers et al., [41].

Borchers, Du [41] determined the k-size-Steiner ratio for graphs exactly: For $k = 2^r + s$, where $0 \leq s < 2^r$, this quantity is

$$m^{(k)}(G) = \frac{r \cdot 2^r + s}{(r+1)\cdot 2^r + s}. \tag{9.30}$$

Karpinski et al. [222] give a tighter analysis of the k-size Steiner ratio. Hourgady, Kirchner [195] improve the the lower bound for the performance ratio.
Although the k-size-Steiner ratio in graphs and the rectilinear plane have been completely determined for $k \geq 2$, this quantity in the Euclidean plane for $k > 2$ is still unknown. Du et al. [126] conjectured that the ratio equals

$$\frac{\sqrt{2}+\sqrt{6}}{1+\sqrt{2}+\sqrt{6}} = 0,79439\ldots.$$

[9] Zelikovsky's algorithm is greedy and works essentially as follows: Start from an MST and at each iteration choose a Steiner point such that using this Steiner point to connect three given point could replace two edges in the MST and such a replacement achieves the maximum length reduction among all possible replacement.

9.7 The Traveling Salesman Problem

The following problem is maybe the well-studied problem in combinatorial optimization. Given: A finite set N of points in a metric space (X, ρ). Find: A cycle $G = (N, E)$ embedded in (X, ρ) such that $L(X, \rho)(G)$ is minimal. In other words, a traveling salesman has to visit the cities of N in arbitrary order and the end he has return to the city from which he started. His goal is to minimize the total length while doing this.
A solution is called a Traveling Salesman Tour (TST) for N. Since we are in a metric space, we may assume that a TST does not pass a vertex more than once. Moreover, the relation to the graph version is easy to see: Additionally points are not necessary and we may understand the problem for N in (X, ρ) as a problem in the complete graph $(N, \binom{N}{2})$, equipped with the length function f defined by $f(\underline{vv'}) = \rho(v, v')$.
Similar to 2.5.11 for $N' \subseteq N$ it holds

$$L(\text{TST for } N') \leq L(\text{TST for } N). \tag{9.31}$$

For the Traveling Salesman Problem a greedy technique is not "ideal" helpful: Consider the four points $v_1 = (0, 3)$, $v_2 = (-8, 3)$, $v_3 = -v_1$ and $v_4 = -v_2$ in the Euclidean plane. A tour obtained by a greedy method is v_1, v_2, v_4, v_3, v_1 of length $22 + 2 \cdot \sqrt{75} = 39.32\ldots$, whereas a TST is given by v_1, v_2, v_3, v_4, v_1 is of length 36.

It is $\mathcal{NP}$-hard to find a TST, compare [162] or [231]. Hence, we are interested in approximation algorithms.[10] Lower bounds can be found by using spanning trees. First observe, that if we have any cycle through the given points and remove one edge then we get a spanning tree. Hence,

$$L(\text{TST for } N) \geq L(\text{MST for } N). \tag{9.32}$$

On the other hand, with the same proof as that given for 3.2.1, we find

$$L(\text{TST for } N) \leq 2 \cdot L(\text{MST for } N). \tag{9.33}$$

These observations create a 2-approximation for an TST in quadratic time: 1. Find an MST T for N; 2. Double every edge of T to obtain an Eulerian graph G; 3. Find an Eulerian cycle in G; 4. Output the tour that visits all vertices of G in the order of their first appearance in the cycle. But, we can do better.

Algorithm 9.7.1 *(Christofides [63]) Let N be a finite set of points in a metric space. Then*

1. *Find an MST $T = (N, E)$ for N;*
2. *Let V be the vertices of T which have odd degree;*
   Compute a perfect matching $M = (V, E')$ with minimal length;[11]

[10] The version that allows dissimilarities is essentially more difficult. Not only is it $\mathcal{NP}$-hard to solve this problem exactly, but also approximately; compare [314].

[11] This can be done with help of a linear programming approach, see [288]. Here Johns lemma 4.5.2 plays also an important role in the so-called ellipsoid algorithm.

3. *Find an Eulerian cycle G in $T \cup M$;*

4. *Create from G a tour that visits the vertices of N in order of their first appearance in G.*

This algorithm runs in cubic time. The performance ratio can determined with help of the following considerations. Recall that the graph G consists of T and M, hence the length of the resulting tour $\underline{G}$ satisfies

$$L(\underline{G}) \leq L(G) = L(T) + L(M). \tag{9.34}$$

Let $V = \{v_1, v_2, \ldots, V_{2m}\}$ be the set of odd-degree vertices in T, in the order that they appear in the shortest tour $\tilde{G}$. Consider the two perfect matchings of V:

$$\begin{aligned} M_1 &= \{\underline{v_1, v_2}, \underline{v_3, v_4}, \ldots, \underline{v_{2m-1}, v_{2m}}\} \text{ and} &(9.35)\\ M_2 &= \{\underline{v_2, v_3}, \underline{v_4, v_5}, \ldots, \underline{v_{2m}, v_1}\}. &(9.36)\end{aligned}$$

By the triangle inequality

$$L(\tilde{G}) \geq L(M_1) + L(M_2) \geq 2 \cdot L(M). \tag{9.37}$$

Substituting (9.32) and (9.37) in (9.34) give

Theorem 9.7.2 *Let N be a finite set of points in a metric space. Then for a result $\underline{G}$ of 9.7.1*

$$L(\underline{G}) \leq \frac{3}{2} \cdot L(\mathit{TST\ for\ N}). \tag{9.38}$$

Finding a better approximation algorithm for the case of a general metric space is currently one of the high-profile open problems in the area of network design, and still unsolved.
For references to the Traveling Salesman Problem and its variations see [34], [115], [179], [218], [231], [288] and [289].

9.8 Shortest multiple-edge-connected Networks

We consider multiple connected graphs: For a positive integer k a k-edge-connected graph is a graph such that for each pair of distinct vertices there are k edge disjoint paths between them.[12] Equivalently, compare [37], a graph $G = (V, E)$ is k-edge-connected if and only if $G = (V, E \setminus E')$ is connected for any set $E' \subseteq E$ of at most $k - 1$ edges. In this sense, a connected graph is 1-edge-connected. A TST is a 2-edge-connected graph.
The problem of finding a shortest graph that multiply connects a finite set of points has applications in the study of fault tolerance of networks. We consider the k-edge-connected Steiner's Network Problem: Given: A finite set N of points in a metric

[12]Note that the degree of each vertex in a k-edge-connected graph is at least k.

space (X, ρ) and an integer $k \geq 1$. Find: A k-edge-connected graph $G = (V, E)$ embedded in (X, ρ) such that $N \subseteq V$; and $L(X, \rho)(G)$ is minimal.
A solution is called a k-edge-connected Steiner Minimal Network (k-edge-SMN) for N. Clearly, a 1-edge-SMN is an SMT for N. A 2-edge-SMN is a network of minimal length containing no bridge, and

$$L(2 - \text{edge-SMN}) \leq L(\text{TST}). \tag{9.39}$$

The edge-connected SMN problem is $\mathcal{NP}$-hard, since it is a generalization of Steiner's Problem.[13]
An approximation is given by the following observations: An MST is an approximation for a 1-edge-SMN and we have an approximation for a 2-edge-SMN, namely for a TST. Consequently,

Algorithm 9.8.1 *Let N be a finite set of points in a metric space. The following algorithm is an approximation for a k-edge-SMN:*

1. *Let $k = 2 \cdot \kappa_1 + \kappa_2$, where $\kappa_2 = 0$ or $= 1$;*
2. *If $\kappa_1 > 0$, then apply 9.7.1 to find T_S which approximates a TST for N;*
3. *If $\kappa_2 \neq 0$, then find an MST T_K for N;*
4. *Construct a spanning multigraph for N consisting of κ_1 duplications of T_S and κ_2 copies of T_K.*

It is easy to verify that this algorithm runs in cubic time. Moreover,

Observation 9.8.2 *(Du, Hu, Jia [139]) For the algorithm $\mathcal{M}_k$ in 9.8.1,*

$$\mathit{error}(\mathcal{M}_k) = \begin{cases} 2 & : \ k \text{ even} \\ 2 + \frac{4}{3k} & : \ \text{otherwise} \end{cases}$$

To discuss this observation more precisely, we compare the length of a k-edge-SMN for N with the length of such a network which does not use Steiner points, a so-called k-edge-connected Minimum Spanning Network (k-edge-MSN) for N:

$$m^{k-\text{edge}}(X, \rho) := \inf \left\{ \frac{L(k - \text{edge-SMN for } N)}{L(k - \text{edge-MSN for } N)} : N \subseteq (X, \rho) \text{ is a finite set} \right\}. \tag{9.40}$$

The quantity $m^{k-\text{edge}}(X, \rho)$ is called the k-connected Steiner ratio. Of course,

$$m^{1-\text{edge}}(X, \rho) = m(X, \rho) \tag{9.41}$$

holds for any metric space (X, ρ). The following facts are essentially deeper:

[13] For the similar problem of constructing a k-edge-connected graph from a given network by adding the minimum number of edges see [52] and [357]. References for approximation algorithms of finding highly connected subgraphs of a network see [226].

Theorem 9.8.3 *(Du, Hu, [138]) In any metric space (X,ρ) it holds for $k \geq 2$*

$$m^{k-edge}(X,\rho) \geq \begin{cases} 1 - \frac{1}{k+2} & : \ k \text{ even} \\ 1 - \frac{1}{k+1} & : \ \text{otherwise} \end{cases}$$

This improved a result by Du, Hu, Jia [139]) that for any $k \geq 2$

$$m^{k-\text{edge}}(X,\rho) \geq \frac{3}{4}. \tag{9.42}$$

Clearly, for more specific spaces we expect better estimates. And indeed, from [139] and [196], we have for $m^{k-\text{edge}}$

k	Euclidean plane	Rectilinear plane
arbitrary	$\geq \frac{\sqrt{3}}{2} = 0.86602\ldots$	
2		$\leq \frac{6}{7} = 0.85714\ldots$
3	$\leq \frac{\sqrt{3}+2}{4} = 0.93301\ldots$	$\leq \frac{7}{8} = 0.875$
4	$\leq \frac{3\sqrt{3}+7}{2\sqrt{3}+9} = 0.97850\ldots$	

9.9 Steiner's Problem in Spaces with a weaker triangle Inequality

Up to now, we have used the triangle inequality as a property of the metric. It is conceivable that slight violations of the triangle inequality should not be too deleterious with respect to performance guarantees of an approximation. Andreae and Bandelt [11] consider the deviation from the triangle inequality captured by a parameter τ in the following relaxation:

$$\rho(v,v') \leq \tau \cdot (\rho(v,w) + \rho(w,v')) \tag{9.43}$$

for all $v, v', w \in X$.
Such a parametrized triangle inequality is given in the situation that the input data are from a fixed range of values. Assume that all distances under consideration are bounded by real numbers L and U in the following way:

$$L \leq \rho(v,v') \leq U \tag{9.44}$$

for different points v and v'.
If $L > 0$, then $\rho(v,w) + \rho(w,v') \geq 2L$, so that $U(\rho(v,w) + \rho(w,v')) \geq 2L\rho(v,v')$. Hence, the metric ρ satisfies the inequality (9.44) with the parameter

$$\tau = \frac{U}{2L} \geq \frac{1}{2}. \tag{9.45}$$

This scenario applies to the Minimum Spanning Tree approximation for Steiner's Problem: When the parameter τ approaches 1/2, the performance guarantee factor 2 decreases and eventually reaches 1; recall 3.2.1. We can see that the factor decreases when we make the additional assumption that, for some τ with $0 < \tau \leq 1$, the set N of given points satisfies the inequality (9.43) for all $v, v' \in N$ and $w \in X \setminus N$.

Theorem 9.9.1 *(Andreae, Bandelt [11]) Let (X, ρ) be a metric space, and let N be a finite subset of X with $|N| = n > 1$. Let $0 < \tau \leq 1$. Suppose that N satisfies equation (9.43) with respect to τ.*
Let T be an SMT and T' be an MST for N in (X, ρ). Then

$$L(T') \leq 2 \cdot \tau \cdot \left(1 - \frac{1}{n}\right) \cdot L(T)$$

if $\tau \geq n/(2n-2)$, and

$$L(T') = L(T)$$

otherwise.

The following example shows that the bound given in 9.9.1 is the best possible: Consider $X = N \cup \{x\}$ with the distances $\rho(v, v') = 2\tau$ for different points v and v', and $\rho(v, x) = 1$.

9.10 The average Case

The Steiner ratio is a quantity which describes a worst-case scenario. On the other hand, the average-case is also of interest. More exactly: Distribute n points $v_1, \ldots, v_n$ by a suitable random process in the space (X, ρ), and then ask for the expected value $E(n) = E(X, \rho)(n)$ of $\mu(\{v_1, \ldots, v_n\})$. Very little is known about these functions. Clearly,

Theorem 9.10.1

$$E(X, \rho)(n) \geq m^n(X, \rho) \geq m(X, \rho). \tag{9.46}$$

Values of $E(n) = E(X, \rho)(n)$ for specific spaces and distributions of points are given by [167], [200] and [366]. In particular

Example 9.10.2 **a)** *(Gilbert, Pollak [167]) In the Euclidean plane it holds:*

$$E(3) \geq 0.94\ldots.$$

b) *(Bern [30][14]) In the plane with rectilinear distance it holds:*

$$\liminf_{n\to\infty} E(n) < 1.$$

[14] which describes also an approximation algorithm

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Index