Anti Lie-Trotter formula

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2015, March (at Greifswald)

(Joint work[1] with K.M.R. Audenaert)

[1] K.M.R. Audenaert and F. Hiai, Anti Lie-Trotter formula, arXiv:1412.7905.

Plan

- Lie-Trotter-Kato formula: survey
- Anti Lie-Trotter formula: $\lim_{r\to\infty}(A^{r/2}B^rA^{r/2})^{1/r}$
- The maximal case
- $\lim_{r\to\infty}(A^r \# B^r)^{2/r}$?

Lie-Trotter-Kato formula: survey

- $\mathcal{H}$ is a Hilbert space,
- H, K are lower bounded self-adjoint operators on $\mathcal{H}$ (not necessarily densely-defined),
- e^{-tH}, e^{-tK} $(t \geq 0)$ are C_0-semigroups on $\mathcal{H}$ (under the convention $e^{-tH} = 0$ on $(\mathrm{dom}\, H)^{\perp}$),
- $\mathcal{D}_0 := \mathrm{dom}\, H_+^{1/2} \cap \mathrm{dom}\, K_+^{1/2}$, $\mathcal{H}_0 := \overline{\mathcal{D}_0}$, P_0 is the projection onto $\mathcal{H}_0$,
- $H_+ \dotplus K_+$ is the form sum, i.e.,

$$\|(H_+ \dotplus K_+)^{1/2}\xi\|^2 = \|H_+^{1/2}\xi\|^2 + \|K_+^{1/2}\xi\|^2, \qquad \xi \in \mathcal{D}_0.$$

- $H \dotplus K := (H_+ \dotplus K_+) - P_0(H_- + K_-)P_0.$

Trotter-Kato formula (Kato, 1978)

$$\lim_{n\to\infty}(e^{-tH/n}e^{-tK/n})^n = \lim_{r\searrow 0}(e^{-rtH/2}e^{-rtK}e^{-rtH/2})^{1/r} = e^{-t(H\dotplus K)}P_0$$

in the strong operator topology, uniformly in $t \in [T_0, T]$, $0 < T_0 < T$.

Refinements of Trotter-Kato formula

(1) Trace norm convergence: Neidhardt-Zagrebnov (1990, 1999), H.[2] (1995)

If $H + K$ is essentially self-adjoint and $e^{-K} \in C_p(\mathcal{H})$ where $0 < p < \infty$, then

$$\lim_{r\searrow 0}\left\|(e^{-rH/2}e^{-rK}e^{-rH/2})^{1/r} - e^{-(H+K)}\right\|_p = 0,$$

$$\lim_{n\to\infty}\left\|(e^{-H/n}e^{-K/n})^{n+1} - e^{-(H+K)}\right\|_p = 0.$$

[2] F. Hiai, Trace norm convergence of exponential product formula, *Lett. Math. Phys.* **33** (1995), 147–158

(2) Operator norm convergence: Rogava (1993), Neidhardt-Zagrebnov (1998, 1999), Ichinose-Tamura (1997, 2001), Ichinose-Neidhardt-Zagrebnov[3] (2004)

Assume that $H, K \geq 0$ and $\mathbf{dom}\, H^{1/2} \cap \mathbf{dom}\, K^{1/2}$ is dense in $\mathcal{H}$.
If $\mathbf{dom}\,((H \dotplus K)^{\alpha}) \subset \mathbf{dom}\, H^{\alpha} \cap \mathbf{dom}\, K^{\alpha}$ for some $\alpha \in (1/2, 1)$ and $\mathbf{dom}\, H^{1/2} \subset \mathbf{dom}\, K^{1/2}$, then

$$\left\| (e^{-tH/2n} e^{-tK/n} e^{-tH/2n})^n - e^{-t(H \dotplus K)} \right\|_{\infty} = O(n^{-(2\alpha-1)}),$$

$$\left\| (e^{-tH/n} e^{-tK/n})^n - e^{-t(H \dotplus K)} \right\|_{\infty} = O(n^{-(2\alpha-1)}),$$

uniformly in $t \in [0, T]$, $0 < T < \infty$, as $n \to \infty$.

Note Trotter-Kato formula $\longleftrightarrow$ Feynman-Kac formula

[3] T. Ichinose, H. Neidhardt and V.A. Zagrebnov, Trotter-Kato product formula and fractional powers of self-adjoint generators, *J. Funct. Anal.* **207** (2004), 33–57.

Lie-Trotter-Kato formula for operator means

Kubo-Ando (1980) For each operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$ the associated operator mean is

$$A \,\sigma\, B := A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

for invertible $A, B \in B(\mathcal{H})^+$, extended to general $A, B \in B(\mathcal{H})^+$ as

$$A \,\sigma\, B := \lim_{\varepsilon \searrow 0} (A + \varepsilon I) \,\sigma\, (B + \varepsilon I).$$

For example,

$\dfrac{A + B}{2}$ arithmetic mean,

$A \,\#\, B$ geometric mean, first introduced by Pusz-Woronowicz,

$\left(\dfrac{A^{-1} + B^{-1}}{2}\right)^{-1}$ harmonic mean.

H.[4] (1997)

Let σ be an operator mean for the corresponding operator monotone function f with $\alpha = f'(1)$. If $H \in B(\mathcal{H})$ is self-adjoint and K is a lower bounded self-adjoint operator on $\mathcal{H}$, then

$$\lim_{r \searrow 0} (e^{-rtH} \,\sigma\, e^{-rtK})^{1/r} = e^{-t((1-\alpha)H + \alpha K)}$$

in the strong operator topology, uniformly in $t \in [0, T]$, $T > 0$.

Matrix case: Audenaert-H. (2014)

For positive semi-definite matrices A, B,

$$\lim_{r \searrow 0} (A^r \,\sigma\, B^r)^{1/r} = P_0 \exp((1-\alpha)\log A \dot{+} \alpha \log B),$$

where $P_0 := A^0 \wedge B^0$.

[4] F. Hiai, Log-majorizations and norm inequalities for exponential operators, in *Linear Operators*, Banach Center Publications, Vol. 38, 1997, pp. 119–181.

Anti Lie-Trotter formula

What happens about

$$\lim_{r\to\infty}(e^{-rH/2}e^{-rK}e^{-rH/2})^{1/r}\ ?$$

Letting $A = e^{-H}$ and $B = e^{-K}$ we may consider

$$\lim_{r\to\infty}(A^{r/2}B^{r}A^{r/2})^{1/r}$$

for $A, B \in B(\mathcal{H})^{+}$.

Motivation from quantum information

For density matrices ρ, σ,

- Rényi relative entropy

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\, \rho^\alpha \sigma^{1-\alpha}$$

- Sandwiched Rényi relative entropy (Müller-Lennert et al., Wilde-Winter-Yang, Frank-Lieb, Beigi, ...)

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\, (\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha = \frac{1}{\alpha - 1} \log \mathrm{Tr}\, (\rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{2}})^\alpha$$

- α-z-relative entropy (Audenaert-Datta)

$$D_{\alpha,z}(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\, (\rho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \rho^{\frac{\alpha}{2z}})^z$$

What are limiting cases for $\alpha \to 0, 1, \infty$ and $z \to 0, \infty$?
Anti Lie-Trotter appears for fixed $\alpha \neq 1$ and $z \to 0$.

Below we assume that $A, B \in B(\mathcal{H})^+$ are compact operators.

- Write

$$A = \sum_{i=1}^{\infty} a_i |v_i\rangle\langle v_i|, \qquad B = \sum_{i=1}^{\infty} b_i |w_i\rangle\langle w_i|$$

  where $a_1 \geq a_2 \geq \dots$ are the eigenvalues of A and $\{v_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ with $Av_i = a_i v_i$.
- $Z_r := (A^{r/2} B^r A^{r/2})^{1/r}$ for $r > 0$.
- $\lambda_1(r) \geq \lambda_2(r) \geq \cdots$ are the eigenvalues of Z_r.

Commuting case If $AB = BA$, then $Z_r = AB$ and $\{w_i\}$ can be a permutation of $\{v_i\}$, so

$$Z_r = \sum_{i=1}^{\infty} a_i b_{j_i} |v_i\rangle\langle v_i|$$

and $(\lambda_i(r))$ is the decreasing rearrangement of $(a_i b_{j_i})$ independently of $r > 0$.

Bourin[5] (2004): Case when A is a projection

Assume that $A = E$ is the projection onto a subspace $\mathcal{E}$ with $\dim\mathcal{E} = k$ and $B = \sum_{i=1}^{d} b_i|w_i\rangle\langle w_i|$ is a positive semi-definite matrix. Then $(EB^rE)^{1/r}$ converges as $r \to \infty$, and if $Ew_1, \dots, Ew_k$ are linearly independent, then

$$\lim_{r\to\infty} \lambda_i((EB^rE)^{1/r}) = b_i \quad \text{for} \quad i = 1, \dots, k.$$

[5] J.-C. Bourin, Convexity or concavity inequalities for Hermitian operators, *Math. Ineq. Appl.* **7** (2004), 607–620.

Fact (Araki[6]-Lieb-Thirring, 1990)

$$(\lambda_i(r))_{i=1}^\infty \prec_{w(\log)} (\lambda_i(r'))_{i=1}^\infty \quad \text{if} \quad 0 < r < r',$$

i.e.,

$$\prod_{i=1}^k \lambda_i(r) \le \prod_{i=1}^k \lambda_i(r')$$

for every $k = 1, 2 \dots$.

Proposition

For every $i = 1, 2, \dots$ the limit

$$\lambda_i := \lim_{r\to\infty} \lambda_i(r)$$

exists, and

$$(\lambda_i(r)) \prec_{w(\log)} (\lambda_i) \prec_{w(\log)} (a_i b_i).$$

[6] H. Araki, On an inequality of Lieb and Thirring, *Lett. Math. Phys.* **19** (1990), 167–170.

Theorem

$Z_r = (A^{r/2}B^r A^{r/2})^{1/r}$ converges in the operator norm as $r \to \infty$.

Lemma 1

$$\lambda_1 = \max\{a_i b_j : \langle v_i, w_j \rangle \neq 0\}$$

- For $k \in \mathbb{N}$, $\mathcal{H}^{\wedge k}$ is the k-fold antisymmetric tensor of $\mathcal{H}$.
- $\mathcal{I}(k)$ is the set of k-tuples $I = (i_1, \dots, i_k)$ in $\mathbb{N}$ with $1 \leq i_1 < \cdots < i_k$.
- For $I \in \mathcal{I}(k)$, $a_I := a_{i_1} \cdots a_{i_k}$, $v_I^\wedge := v_{i_1} \wedge \cdots \wedge v_{i_k} \in \mathcal{H}^{\wedge k}$.
- Note: $\langle v_I^\wedge, w_J^\wedge \rangle = \det[\langle v_i, w_j \rangle]$.

Lemma 2

For every $k \in \mathbb{N}$,

$$\lambda_1 \lambda_2 \cdots \lambda_k = \max\{a_I b_J : I, J \in \mathcal{I}_d(k), \langle v_I^\wedge, w_J^\wedge \rangle \neq 0\},$$

Lemma 3

For any $k \in \mathbb{N}$ there are constants $\alpha, \beta > 0$ (depending on only k) such that

$$\alpha \|P - Q\|_\infty \le \inf_{\theta \in \mathbb{R}} \left\| \phi_1 \wedge \cdots \wedge \phi_k - e^{\sqrt{-1}\theta} \psi_1 \wedge \cdots \wedge \psi_k \right\| \le \beta \|P - Q\|_\infty$$

for every orthonormal $\{\phi_1, \ldots, \phi_k\}$, $\{\psi_1, \ldots, \psi_k\} \subset \mathcal{H}$, the projections P onto $\mathrm{span}\{\phi_1, \ldots, \phi_k\}$ and Q onto $\mathrm{span}\{\psi_1, \ldots, \psi_k\}$.

- Φ is a symmetric gauge function.
- $\|X\|_\Phi := \Phi(\mu_1(X), \mu_2(X), \dots)$ for $X \in B(\mathcal{H})$, where $\mu_1(X) \ge \mu_2(X) \ge \cdots$ are the singular values of X.
- $C_\Phi(\mathcal{H})$ is the corresponding symmetrically normed ideal, i.e.,

$$C_\Phi(\mathcal{H}) := \{X \in B(\mathcal{H}) : \|X\|_\Phi < +\infty\}.$$

- Assume that Φ is regular: $C_\Phi(\mathcal{H}) = C_\Phi^{(0)}(\mathcal{H})$.

Theorem

Assume $\Phi(a_1b_1, a_2b_2, \dots) < +\infty$. Then $Z_\infty = \lim_{r\to\infty} Z_r$ is in $C_\Phi(\mathcal{H})$ and

$$\|Z_r - Z_\infty\|_\Phi \to 0 \quad \text{as} \quad r \to \infty.$$

Corollaries

(1) If either A or B is in $\mathcal{C}_\Phi(\mathcal{H})$, then $Z_\infty \in \mathcal{C}_\Phi(\mathcal{H})$ and

$$\|Z_r - Z_\infty\|_\Phi \to 0 \quad \text{as} \quad r \to \infty.$$

(2) Let $1 \le p, p_1, p_2 \le \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. If $A \in \mathcal{C}_{p_1}(\mathcal{H})$ and $B \in \mathcal{C}_{p_2}(\mathcal{H})$, then $Z_\infty \in \mathcal{C}_p(\mathcal{H})$ and

$$\|Z_r - Z_\infty\|_p \to 0 \quad \text{as} \quad r \to \infty.$$

The maximal case

Below assume that A, B are positive semi-definite $d \times d$ matrices (though the compact operator case is also valid).

- Choose $0 = i_0 < i_1 < \cdots < i_{m-1} < i_m = d$ and $0 = j_0 < j_1 < \cdots < j_{l-1} < j_l = d$ so that

$$a_1 = \cdots = a_{i_1} > a_{i_1+1} = \cdots = a_{i_2} > \cdots > a_{i_{m-1}+1} = \cdots = a_{i_m},$$
$$b_1 = \cdots = b_{j_1} > b_{j_1+1} = \cdots = b_{j_2} > \cdots > b_{j_{l-1}+1} = \cdots = b_{j_l}.$$

- $\mathcal{I}_d(k) := \{(i_1, \dots, i_k) : 1 \le i_1 < \cdots < i_k \le d\}$ for $k = 1, \dots, d$.
- Write

$$A = \sum_{i=1}^{d} a_i |v_i\rangle\langle v_i| = V\mathrm{diag}(a_1, \dots, a_d)V^* \quad (a_1 \ge \cdots \ge a_d),$$

$$B = \sum_{i=1}^{d} b_i |w_i\rangle\langle w_i| = W\mathrm{diag}(b_1, \dots, b_d)W^* \quad (b_1 \ge \cdots \ge b_d),$$

Theorem

The following are equivalent:

(i) $(\lambda_i)_{i=1}^d = (a_i b_i)_{i=1}^d$;

(ii) for every $k = 1, \ldots, d$ with $i_{r-1} < k \le i_r$ and $j_{s-1} < k \le j_s$, there are $I_k, J_k \in \mathcal{I}_d(k)$ such that $\{1, \ldots, i_{r-1}\} \subset I_k \subset \{1, \ldots, i_r\}$, $\{1, \ldots, j_{s-1}\} \subset J_k \subset \{1, \ldots, j_s\}$ and $\det[\langle v_i, w_j\rangle]_{i\in I_k, j\in J_k} \ne 0$;

(iii) the property in (ii) holds for every $k = i_r$ or j_s up to $\min\{i_{m-1}, j_{l-1}\}$.

Corollaries

(1) If $\det[\langle v_i, w_j\rangle]_{i\in I, j\in J} \ne 0$ for all $I, J \subset \{1, \ldots, d\}$ with $|I| = |J|$ (i.e., all minors of V^*W are non-zero), then $(\lambda_i)_{i=1}^d = (a_i b_i)_{i=1}^d$.

Corollaries (cont.)

(2) Assume $a_1 > \cdots > a_d$ and $b_1 > \cdots > b_d$. Then $(\lambda_i)_{i=1}^d = (a_i b_i)_{i=1}^d$ if and only if $\det[\langle v_i, w_j \rangle]_{1 \le i,j \le k} \ne 0$ for all $k = 1, \dots, d$.

(3) Assume $a_1 > \cdots > a_d$ and the conditions of the theorem hold. Then

$$\lim_{r \to \infty} Z_r = V \mathrm{diag}(a_1 b_1, \dots, a_d b_d) V^*.$$

(4) (Bourin's case) When $A = E$ is the orthogonal projection onto a subspace $\mathcal{E}$ with $\dim \mathcal{E} = k$, $\lim_{r \to \infty} \lambda_i((EB^rE)^{1/r}) = b_i$ for $i = 1, \dots, k$ if and only if $Ew_1, \dots, Ew_k$ are linearly independent.

Question

$$(\lambda_i)_{i=1}^d = (a_1 b_{j_1}, \dots, a_d b_{j_d})^{\downarrow}$$

for some permutation $(j_1, \dots, j_d)$? Confirmed to be true for $d \le 5$.

Extension to more than two matrices

Theorem

For every $d \times d$ positive semi-definite matrices $A_1, \dots, A_m$,

$$(A_1^{r/2} A_2^{r/2} \cdots A_{m-1}^{r/2} A_m^r A_{m-1}^{r/2} \cdots A_2^{r/2} A_1^{r/2})^{1/r}$$

converges as $r \to \infty$.

In typical situation, the limit eigenvalue vector is

$$(a_1^{(1)} a_1^{(2)} \cdots a_1^{(m)}, a_2^{(1)} a_2^{(2)} \cdots a_2^{(m)}, \dots, a_d^{(1)} a_d^{(2)} \cdots a_d^{(m)}),$$

where $a_1^{(k)} \geq \cdots \geq a_d^{(k)}$ are the eigenvalues of A_k.

$\lim_{r\to\infty}(A^r \# B^r)^{2/r}$?

Fact (Kato,[7] 1979)

For arithmetic mean,

$$A \vee B := \lim_{r\to\infty}\left(\frac{A^r + B^r}{2}\right)^{1/r} = \lim_{r\to\infty}(A^r + B^r)^{1/r}.$$

Also, for harmonic mean,

$$A \wedge B := \lim_{r\to\infty}(A^r \,!\, B^r)^{1/r}.$$

What about $\lim_{r\to\infty}(A^r \# B^r)^{2/r}$ for geometric mean $\#$?

[7] T. Kato, Spectral order and a matrix limit theorem, *Linear and Multilinear Algebra* **8** (1979), 15–19.

Fact (Ando-H.,[8] 1994)

For $G_r := (A^r \# B^r)^{2/r}$,

$$(\lambda_i(G_r))_{i=1}^d \succ_{(\log)} (\lambda_i(G_{r'}))_{i=1}^d \quad \text{if} \quad 0 < r < r'.$$

Proposition

For every $i = 1, \ldots, d$ the limit

$$\hat{\lambda}_i := \lim_{r \to \infty} \lambda_i(G_r)$$

exists, and

$$(a_i b_{d+1-i})_{i=1}^d \prec_{(\log)} (\hat{\lambda}_i)_{i=1}^d \prec_{(\log)} (\lambda_i(G_r))_{i=1}^d.$$

[8] T. Ando and F. Hiai, Log majorization and complementary Golden-Thompson type inequalities, *Linear Algebra Appl.* **197/198** (1994), 113–131.

Proposition: 2×2 case

- $$\lim_{r\to\infty} G_r = (\alpha\beta)^{1/2} \frac{((\alpha^{-1/2}A) \vee (\beta^{-1/2}B))^2}{\det((\alpha^{-1/2}A) \vee (\beta^{-1/2}B))},$$

  where $\alpha := \det A$ and $\beta := \det B$.

- If $\langle v_1, w_2\rangle = (V^*W)_{12} = 0$ (i.e., V^*W is diagonal), then

  $$(\lambda_1(G_r), \lambda_2(G_r)) = (a_1b_1, a_2b_2).$$

- If $(V^*W)_{12} \neq 0$, then

  $$\lim_{r\to\infty}(\lambda_1(G_r), \lambda_2(G_r)) = (a_1b_2, a_2b_1)^{\downarrow}.$$

Question

- Does $G_r = (A^r \# B^r)^{2/r}$ converge as $r \to \infty$?
- In the typical situation

$$\lim_{r\to\infty} (\lambda_i(G_r))_{i=1}^d = (a_1 b_d, a_2 b_{d-1}, \dots, a_d b_1)^{\downarrow} \,?$$

In place of summary

$$
\begin{aligned}
& A^{\downarrow} B^{\downarrow} \\
& \succ_{(\log)} Z_r = (A^{r/2} B^r A^{r/2})^{1/r} \quad \begin{matrix} \uparrow & (r \to \infty) & \text{typically} \\ \downarrow & (r \to 0) & \end{matrix} \\
& \succ_{(\log)} P_0 \exp(\log A \dotplus \log B) \\
& \succ_{(\log)} G_r = (A^r \# B^r)^{2/r} \quad \begin{matrix} \uparrow & (r \to 0) & \\ \downarrow & (r \to \infty) & \text{typically ?} \end{matrix} \\
& \succ_{(\log)} A^{\downarrow} B^{\uparrow}
\end{aligned}
$$

Thank you for your attention.