Positive Lévy processes in classical and free probability

April 6, 2015

Takahiro Hasebe (Hokkaido Univ.)

Aim

- Positive Lévy processes,
- A failure of "functoriality" (= analogy, transfer principle) between classical and free
- What is a free gamma distribution?

My colleague Hayato Saigo says "mathematical definition of 'analogy' is a functor".

Lévy processes

$\mathcal{L}(X)$ denotes the law of a random variable X.

Definition

$\mathbb{R}$-valued $(X_t)_{t\geq 0}$ is called a Lévy process iff

- $X_0 = 0$,
- *For all* $0 = t_0 < t_1 < \cdots < t_n$,
  $X_{t_n} - X_{t_{n-1}}, X_{t_{n-1}} - X_{t_{n-2}}, \ldots, X_{t_1} - X_{t_0}$ *are indep.,*
- *For all* $0 \leq s < t$, $\mathcal{L}(X_t - X_s) = \mathcal{L}(X_{t-s})$.
- $t \mapsto X_t$ *is right continuous with left limit,*
- *The map* $t \mapsto \mathcal{L}(X_t)$ *is weakly continuous.*

Free Lévy processes

$(\mathcal{A}, \tau)$: II_1 factor, τ: normal faithful finite trace.

Definition

Let a map $t \mapsto X_t$ take values in selfadjoint operators affiliated to $\mathcal{A}$. $(X_t)_{t \geq 0}$ is called a free Lévy process iff

- $X_0 = 0$,
- *For all* $0 = t_0 < t_1 < \cdots < t_n$, $X_{t_n} - X_{t_{n-1}}, X_{t_{n-1}} - X_{t_{n-2}}, \ldots, X_{t_1} - X_{t_0}$ *are free indep.*,
- *For all* $0 \leq s < t$, $\mathcal{L}(X_t - X_s) = \mathcal{L}(X_{t-s})$.
- *The map* $t \mapsto \mathcal{L}(X_t)$ *is weakly continuous.*

Infinitely divisible distribution

Definition

A probability measure μ *on* $\mathbb{R}$ *is said to be infinitely divisible iff for all* $n \in \mathbb{N}$ *there exists* μ_n *such that* $\mu = \underbrace{\mu_n * \cdots * \mu_n}_{n \text{ fold}}$.

Proposition

A probability measure μ *is ID iff there exists a Lévy process* (X_t) *such that* $\mathcal{L}(X_1) = \mu$.

Proof.

($\Leftarrow$) Given a Lévy process (X_t), let $\mu^{*t} := \mathcal{L}(X_t)$. Then $\mu^{*t} * \mu^{*s} = \mu^{*(t+s)}$. So let $\mu_n := \mu^{*1/n}$.
($\Rightarrow$) Define $\mu^{*1/n} := \mu_n$, and define $\mu^{*m/n} := (\mu_n)^{*m}$. Then extend the parameter m/n to real numbers by continuity $\Rightarrow$ convolution semigroup $(\mu^{*t})_{t \geq 0}$. Then construct a Lévy process. $\square$

Freely infinitely divisible distribution

If X, Y are free indep. selfadjoint operators affiliated to $(\mathcal{A}, \tau)$, then $\mathcal{L}(X+Y)$ is denoted as $\mathcal{L}(X) \boxplus \mathcal{L}(Y)$

Definition

A probability measure μ on $\mathbb{R}$ is said to be freely infinitely divisible iff for all $n \in \mathbb{N}$ there exists μ_n such that $\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ fold}}$.

Proposition (Biane '98, Barndorff-Nielsen & Thorbjørnsen '02)

A probability measure μ is FID iff there exists a free Lévy process (X_t) such that $\mathcal{L}(X_1) = \mu$.

Subordinator

Definition

A Lévy process $(X_t)_{t\geq 0}$ *is called a subordinator iff* $t \mapsto X_t$ *is non-decreasing*
$\Leftrightarrow X_t \geq 0$ *for all* $t > 0$
$\Leftrightarrow \mathcal{L}(X_t)$ *is supported on* $[0,\infty)$ *for all* $t > 0$

Definition (Arizmendi-Sakuma-T.H. '13)

A Lévy process $(X_t)_{t\geq 0}$ *is called a free subordinator iff* $t \mapsto X_t$ *is non-decreasing for all* $t > 0$
$\Leftrightarrow X_t \geq 0$ *for all* $t > 0$
$\Leftrightarrow \mathcal{L}(X_t)$ *is supported on* $[0,\infty)$ *for all* $t > 0$

Why "subordinator"?

- In complex analysis: a (analytic) function f is subordinated to g if $\exists h$ s.t. $f = g \circ h$.
- In free probability: for X, Y: free, a subordination function (= subordinator) for free convolution is a function F such that

$$G_{X+Y}(z) = G_X(F(z)).$$

- In classical probability, subordination is a random time change: If $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ are Lévy processes, $Y_t \geq 0$, then $Z_t := X_{Y_t}$ is again a Lévy process.
  Note: $C_{Z_1}(u) = C_{Y_1} \circ C_{X_1}(u)$, where $C(u)$ is cumulant generating function.

Time evolution

Theorem (See Sato's book)

Let $(X_t)_{t\geq 0}$ be a Lévy process. TFAE:

1. *(X_t) is a subordinator, i.e. $\mathcal{L}(X_t)$ is supported on $[0,\infty)$ for all $t>0$.*
2. *$\mathcal{L}(X_t)$ is supported on $[0,\infty)$ for some $t>0$.*
3. *$\mathcal{L}(X_1)$ is supported on $[0,\infty)$.*

Therefore,

$$\begin{aligned}\{\text{the laws of subordinators}\} &= \{\mu \in \mathrm{ID} \mid \mathrm{supp}(\mu^{*t}) \subset [0,\infty), \forall t>0\} \\ &= \{\mu \in \mathrm{ID} \mid \mathrm{supp}(\mu) \subset [0,\infty)\}\end{aligned}$$

Failure of the free analogue

The free analogue fails to hold.

Example

Let (X_t) be a free Lévy process such that $\mathcal{L}(X_1)$ is the shifted semicircle law $\frac{\sqrt{4-(x-2)^2}}{2\pi} 1_{[0,4]}(x)\, dx$. Then $\mathrm{supp}\mathcal{L}(X_t) \not\subset [0,\infty)$ for $t \in (0,1)$.

Hence

$$\begin{aligned}
&\{\text{the laws of free subordinators}\} \\
&\quad = \{\mu \in \mathrm{FID} \mid \mathrm{supp}(\mu^{\boxplus t}) \subset [0,\infty), \forall t > 0\} \\
&\quad \neq \{\mu \in \mathrm{FID} \mid \mathrm{supp}(\mu) \subset [0,\infty)\}
\end{aligned}$$

Monotone, Boolean time evolution

Theorem (T.H. '10)

Let $(X_t)_{t\geq 0}$ be a monotone/Boolean Lévy process. TFAE:

1. (X_t) *is a monotone/Boolean subordinator, i.e.* $\mathcal{L}(X_t)$ *is supported on* $[0,\infty)$ *for all* $t > 0$.
2. $\mathcal{L}(X_t)$ *is supported on* $[0,\infty)$ *for some* $t > 0$.
3. $\mathcal{L}(X_1)$ *is supported on* $[0,\infty)$.

Therefore,
$\{$the laws of Boolean/monotone subordinators$\}$
$= \{\mu \in \text{all probability measures/MID} \mid \mathrm{supp}(\mu) \subset [0,\infty)\}$

Lévy-Khintchine representation

Proposition

μ is ID iff

$$\int_{\mathbb{R}} e^{\mathrm{i}ut}\mu(dt) = \exp\left(\mathrm{i}\eta u - \frac{1}{2}au^2 + \int_{\mathbb{R}\setminus\{0\}}(e^{\mathrm{i}ut} - 1 - \mathrm{i}ut\,1_{[-1,1]}(t))\nu(dt)\right)$$

for some $\eta \in \mathbb{R}$, $a \geq 0$ and ν such that $\int_{\mathbb{R}} \min(1, t^2)\nu(dt) < \infty$.

Lévy-Khintchine representation on $[0, \infty)$

Proposition

μ is ID and supported on $[0, \infty)$ iff

$$\int_{\mathbb{R}} e^{\mathrm{i}ut}\mu(dt) = \exp\left(\mathrm{i}\eta' u + \int_{(0,\infty)} (e^{\mathrm{i}ut} - 1)\nu(dt)\right)$$

for some $\eta' \geq 0$ and ν such that $\int_{(0,\infty)} \min(1, t)\nu(dt) < \infty$.

Free Lévy-Khintchine representation

Let $G_X(z) = \tau((z - X)^{-1})$ and $\mathcal{R}_X(z) = zG_X^{-1}(z) - 1$. If X, Y are free indep. then

$$\mathcal{R}_{X+Y}(z) = \mathcal{R}_X(z) + \mathcal{R}_Y(z).$$

Rem: $\mathcal{R}_X(z) = \sum_{n=1}^{\infty} R_n(X) z^n$.

Proposition

μ is FID iff

$$\mathcal{R}_\mu(z) = \eta z + az^2 + \int_{\mathbb{R}\setminus\{0\}} \left(\frac{1}{1 - zt} - 1 - zt\, 1_{[-1,1]}(t) \right) \nu(dt)$$

for some $\eta \in \mathbb{R}$, $a \geq 0$ and ν such that $\int_{\mathbb{R}} \min(1, t^2)\nu(dt) < \infty$.

Free Lévy-Khintchine representation for free subordinators

Proposition

Suppose μ is FID. Then μ is the law of a free subordinator (i.e. $\mu^{\boxplus t}$ is supported on $[0, \infty)$ for all $t > 0$) iff

$$\mathcal{R}_\mu(z) = \eta' z + \int_{(0,\infty)} \left(\frac{1}{1 - tz} - 1 \right) \nu(dt)$$

for some $\eta' \geq 0$ and ν such that $\int_{(0,\infty)} \min(1, t)\nu(dt) < \infty$.

Rem: In terms of Bercovici-Pata bijection, the laws of classical subordinators are in one-to-one correspondence with the laws of free subordinators.

Examples of laws of subordinators

- Poisson distributions
- Gamma distributions: $\gamma(p,\theta)(dx) = C \cdot x^{p-1} e^{-x/\theta} 1_{(0,\infty)}(x)\, dx$
- Positive stable distributions: $\int_{(0,\infty)} e^{-ux} \mathbf{n}_\alpha(dx) = \exp(-u^\alpha)$ $(\alpha \in (0,1))$.
- Free Poisson distributions $\mathcal{R}_{\pi_t}(z) = \frac{tz}{z-1}$:

$$\pi_t(dx) = \max\{1-t, 0\}\delta_0 + \frac{\sqrt{((1+\sqrt{t})^2 - x)(x-(1-\sqrt{t})^2)}}{2\pi x} 1_{((1-\sqrt{t})^2,(1+\sqrt{t})^2)}(x)\,\mathrm{d}x.$$

- Boolean stable law: $\mathbf{b}_\alpha(dx) = \frac{\sin(\alpha\pi)}{\pi} \cdot \frac{x^{\alpha-1}}{x^{2\alpha}+2\cos(\alpha\pi)x^\alpha+1} 1_{(0,\infty)}(x)\, dx$, $\alpha \in (0, \frac{1}{2}]$.
- Positive free stable distributions $\mathbf{f}_\alpha$: $\mathcal{R}_{\mathbf{f}_\alpha}(z) = -(-z)^\alpha$ $(\alpha \in (0,1))$.

Generalized gamma convolutions

GGC is the weak closure of

$$\{\delta_a * \gamma(p_1, \theta_1) * \cdots * \gamma(p_n, \theta_n) \mid a, p_i, \theta_i > 0, n \in \mathbb{N}\} \subset \mathrm{ID}(\mathbb{R}_+)$$

GGC was introduced by Thorin '77, who is famous for interpolation theory. The motivation for introducing GGC was to prove the infinite divisibility of the log normal distribution $\frac{x^{-1}}{\sqrt{2\pi\sigma^2}} e^{-(\log x - m)^2/2\sigma^2} 1_{(0,\infty)}(x)\, dx$ (the law of e^X when $X \sim N(m, \sigma^2)$).

Example

- Positive stable laws,
- beta distributions of 2nd kind $\frac{1}{B(p,q)} \cdot \frac{x^{p-1}}{(1+x)^{p+q}} 1_{(0,\infty)}(x)\, dx$,
- measures of the form $C \cdot x^{\beta-1} \prod_{i=1}^{n} \frac{1}{(1+\gamma_i x)^{p_i}} 1_{(0,\infty)}(x)\, dx$.

An application of GGC method: If $X \sim N(0,1)$ then $X^n \in \mathrm{ID}$ for $n \in \mathbb{N}$.

Properties of GGC

Theorem (Bondesson '13)

If $X, Y \in GGC$ and indep. then $XY \in GGC$.

The proof is not easy. This problem had been open for $20 \sim 30$ years.

Theorem (Bondesson '13)

If $X \in GGC$ then $e^X \in GGC$.

Conjecture

If $X \in GGC$ and $p \geq 1$, then $X^p \in GGC$.

This conjecture was proved for some subclasses of GGC.
If this conjecture is true, then $X \in \mathrm{GGC} \Rightarrow 1 + \frac{X}{p} \in \mathrm{GGC} \Rightarrow (1 + \frac{X}{p})^p \in \mathrm{GGC} \Rightarrow e^X \in \mathrm{GGC}$.

Question by Bondesson

Is there a set of probability measures $\mathcal{P}$ on $[0, \infty)$ such that

- If $X \in \mathcal{P}$ and $c > 0$ then $cX \in \mathcal{P}$.
- If $X, Y \in \mathcal{P}$ are indep. then $X + Y, XY \in \mathcal{P}$.
- $\mathcal{P}$ is closed wrt the weak convergence.

GGC is only the nontrivial example of such $\mathcal{P}$ ("trivial" means that $\mathcal{P} = \{\delta_0\}$ or $\mathcal{P}$ is the set of all probability measures on $[0, \infty)$).

Bondesson's question in the free probability setting

Consider the free analogue of Bondesson's question:

- If $X \in \mathcal{P}$ and $c > 0$ then $cX \in \mathcal{P}$.
- If $X, Y \in \mathcal{P}$ are free indep. then $X + Y, \sqrt{X}Y\sqrt{X} \in \mathcal{P}$.
- $\mathcal{P}$ is closed wrt the weak convergence.

Is there such a nontrivial class $\mathcal{P}$? $\Rightarrow$ Yes!

Let FS be the set of distributions of free subordinators, i.e.

$$\mathrm{FS} = \{\mu \in \mathrm{FID} \mid \mathrm{supp}(\mu^{\boxplus t}) \subset [0, \infty), \forall t > 0\}$$

Theorem (Arizmendi-Sakuma-T.H. '13)

If $X, Y \in FS$ and free indep. then $X + Y, \sqrt{X}Y\sqrt{X} \in FS$.

Proof.

Use the identity $(\mu \boxtimes \nu)^{\boxplus t} = D_{1/t}(\mu^{\boxplus t} \boxtimes \nu^{\boxplus t})$. $\square$

What is a free gamma distribution?

- The 1st definition (Bożejko-Bryc '06): Free gamma distributions as a special case of free Meixner distributions.
- The 2nd definition (Haagerup & Thorbjørnsen '14): The classical gamma distribution has LK representation

$$\int_{(0,\infty)} e^{\mathrm{i}xu}\gamma(p,1)(dx) = \exp\left(p\int_0^\infty (e^{\mathrm{i}ux}-1)\frac{e^{-x}}{x}\,dx\right)$$

  In terms of Bercovici-Pata bijection we may define

$$\mathcal{R}_{\text{free gamma}}(z) = p\int_0^\infty \left(\frac{1}{1-xz}-1\right)\frac{e^{-x}}{x}\,dx.$$

- The 3rd definition: free gamma = free Poisson. Why?

Free Poisson distribution = free gamma distribution

Theorem (Lukacs '55)

Suppose X, Y are indep. Then X, Y have gamma distributions $\gamma(p_1, \theta), \gamma(p_2, \theta)$ iff $\frac{X}{X+Y}, X+Y$ are indep.

Theorem (Szpojankowski)

Suppose X, Y are free indep. Then X, Y have free Poisson distributions $\pi(p_1, \theta), \pi(p_2, \theta)$ such that $p_1 + p_2 > 1$ "iff" $(X+Y)^{-1/2} X (X+Y)^{-1/2}, X+Y$ are free indep.

Rem: $\mathcal{R}_{\pi(p,\theta)}(z) = \frac{p\theta z}{1-\theta z} = \int_0^\infty \left(\frac{1}{1-xz} - 1 \right) p\delta_\theta(dx)$, $p, \theta > 0$.

Another fact supporting the definition:

- $X \sim N(0,1) \Rightarrow X^2 \sim \gamma(1/2, 2)$.
- $X \sim S(0,1) \Rightarrow X^2 \sim \pi(1,1)$.

Free Poisson distribution = free gamma distribution

$\circledast$: classical multiplicative convolution
$\boxtimes$: free multiplicative convolution

Theorem (Steutel '70, '80, Kristiansen '94)

Let $p \leq 2$. Then for any μ on $[0,\infty)$ we have

$$\mu \circledast \gamma(p,1) \in ID.$$

$p = 2$ is optimal.

Theorem (Arizmendi et al. '10, Perez-Abreu & Sakuma '12)

Let $p \leq 1$. Then for any μ on $[0,\infty)$ we have

$$\mu \boxtimes \pi(p,1) \in FS.$$

GFGC (Generalized Free Gamma Convolution)

GFGC:= the weak closure of

$$\{\delta_a \boxplus \pi(p_1, \theta_1) \boxplus \cdots \boxplus \pi(p_n, \theta_n) \mid a, p_i, \theta_i > 0, n \in \mathbb{N}\} \subset \mathrm{FS}.$$

Actually

$$\mathrm{GFGC} = \mathrm{FS}$$

because:

- Recall

$$\mathcal{R}_{\pi(p,\theta)}(z) = \frac{p\theta z}{1-\theta z} = \int_0^\infty \left(\frac{1}{1-xz} - 1\right) p\delta_\theta(dx);$$

- $\mu \in \mathrm{FS}$ is characterized by free LK

$$\mathcal{R}_\mu(z) = az + \int_0^\infty \left(\frac{1}{1-xz} - 1\right) \nu(dx).$$

Bondesson's conjecture for the free case

Question

If $X \in \mathrm{FS}$ *and* $p \geq 1$*, is it true that* $X^p \in \mathrm{FS}$*?*

Question

If $X \in \mathrm{FS}$*, is it true that* $e^X \in \mathrm{FS}$*?*

Applications of FS

Theorem (Arizmendi-Sakuma-T.H. '13)

- *If $X, Y \in FID$ are free, then $\mathrm{i}(XY - YX) \in FID$.*
- *If $\mathcal{L}(X)$ is symmetric (around 0), then*

$$X \in FID \Leftrightarrow \mathcal{L}(X^2) = \pi(1,1) \boxtimes \sigma \text{ for some } \sigma \in FS.$$

Problem

Find a polynomial $P(X, Y)$ which preserves the FID property.

Proposition (Arizmendi-Sakuma-T.H. '13, Lehner, Chistyakov)

If $X \sim S(0,1)$ then $X^2, X^4, X^6 \in FID$.

Conjecture

If $X \sim S(0,1)$ then $X^n \in FID$ for all $n \in \mathbb{N}$.

Boolean stable law has strong ID property

Definition (Speicher-Woroudi '97)

A positive Boolean stable law $\mathbf{b}_\alpha$ *is defined by*

$$G_{\mathbf{b}_\alpha}(z) = \frac{1}{z - (-z)^{1-\alpha}}, \quad \alpha \in (0, 1]$$

$\circledast$: classical multiplicative convolution

Theorem (Arizmendi-T.H.)

Suppose $\alpha \le \frac{1}{2}$*. Then* $\mu \circledast \mathbf{b}_\alpha \in FS \cap ID$ *for any probability measure* μ *on* $[0, \infty)$*.*

Strange identity

$\circledast$: classical multiplicative convolution
$\boxtimes$: free multiplicative convolution
μ^p: the law of X^p when $\mathcal{L}(X) = \mu$

Theorem (Arizmendi-T.H.)

Suppose $\alpha \in (0,1)$. *Then*

$$\mu^{1/\alpha} \circledast \mathbf{b}_\alpha = \mu^{\boxtimes 1/\alpha} \boxtimes \mathbf{b}_\alpha$$

for any probability measure μ *on* $[0,\infty)$.

Proof.

We show that $G_{\mu^{1/\alpha} \circledast \mathbf{b}_\alpha}(z) = -(-z)^{\alpha-1} G_\mu(-(-z)^\alpha)$. Then $\eta_{\mu^{1/\alpha} \circledast \mathbf{b}_\alpha}(z) = \eta_\mu(-(-z)^\alpha)$. Then $\Sigma_{\mu^{1/\alpha} \circledast \mathbf{b}_\alpha}(z) = \frac{1}{z}\eta^{-1}_{\mu^{1/\alpha} \circledast \mathbf{b}_\alpha}(z) = \Sigma_\mu(z)^{1/\alpha}(-z)^{\frac{1-\alpha}{\alpha}} = \Sigma_{\mu^{\boxtimes 1/\alpha}}(z)\Sigma_{\mathbf{b}_\alpha}(z)$. $\square$

Examples of FID

Theorem (Belinschi-Bożejko-Lehner-Speicher '11)

The normal law $N(0,1)$ is FID.

Theorem (T.H. '14)

- *Beta distribution* $C \cdot x^{p-1}(1-x)^{q-1}1_{(0,1)}(x)\,dx \in$ *FID if* $(p,q) \in D$,
- *Beta distribution of the 2nd kind:* $C \cdot \frac{x^{p-1}}{(1+x)^{p+q}}1_{(0,\infty)}(x)\,dx \in$ *FID if* $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \infty)$.

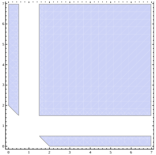

Figure : The region D

Corollary

- *$\gamma(p,\theta)$: $C \cdot x^{p-1} e^{-x/\theta}\, 1_{(0,\infty)}(x)\, dx \in$ FID for $p \in (0, \frac{1}{2}] \cup [\frac{3}{2}, \infty)$*
- *Inverse gamma: $C \cdot x^{-p-1} e^{-\theta/x}\, 1_{(0,\infty)}(x)\, dx \in$ FID for all $p > 0$.*

Rem: $X \sim \gamma(p,\theta) \Rightarrow 1/X \sim$ inverse gamma

Free selfdecomposable distributions

$D_c\mu$ is the law of cX when $\mathcal{L}(X) = \mu$.

Definition

A measure μ on $\mathbb{R}$ is selfdecomposable (SD), if

$$\forall c \in (0,1)\ \exists \mu_c \text{ s.t. } \mu = D_c\mu * \mu_c.$$

Rem: μ and μ_c are necessarily ID.

Definition

A measure μ on $\mathbb{R}$ is FSD if

$$\forall c \in (0,1)\ \exists \mu_c \text{ s.t. } \mu = D_c\mu \boxplus \mu_c.$$

Rem: μ and μ_c are necessarily FID.

Unimodality

Definition

A measure μ on $\mathbb{R}$ is called unimodal, if, for some a in $\mathbb{R}$, it has the form

$$\mu = \mu(\{a\})\delta_a + f(x)\,\mathrm{d}x,$$

where f is increasing on $(-\infty, a)$ and decreasing on (a, ∞).

Theorem (Yamazato '78)

All SD distributions are unimodal.

Theorem (Thorbjørnsen-T.H. '15)

All FSD distributions are unimodal.

Open problems

- Show that if $X \in \text{GGC}$ and $p \geq 1$, then $X^p \in \text{GGC}$.
- Show that if $X \in \mathrm{FS}$ and $p \geq 1$, then $X^p \in \mathrm{FS}$. If this is true, then $\mathcal{L}(X^{q/2}) = \mathcal{L}(|Y|^q) \in \mathrm{FS}$ for $X \sim \pi(1,1)$, $Y \sim S(0,1)$, $q \geq 2$.
- Show that if $X \sim S(0,1)$ then $X^n \in \text{FID}$ for all $n \in \mathbb{N}$.
- Find a selfadjoint polynomial $P(X, Y)$ which preserves the FID property.
- Positive Lévy processes in quantum groups?