Vertex Operator Algebras, Geometry and Mathematical Physics

Seminar Description

This seminar provides an introduction to the theory of Vertex Operator Algebras (VOAs), moving from their algebraic foundations to their geometric realization as sheaves on manifolds (Chiral de Rham Complex) and analytic applications in CFT (KZ equations and modularity).

Main References

[FBZ] E. Frenkel, D. Ben-Zvi: *Vertex Algebras and Algebraic Curves.*

[LL] J. Lepowsky, H. Li: *Introduction to Vertex Operator Algebras and Their Representations.*

[MSV] Malikov, Schechtman, Vaintrob: *Chiral de Rham complex.*

Phase 1: Axiomatics and Fundamental Constructions

This phase establishes the formal framework and the abstract definitions according to [LL], followed by the verification of these structures on the central examples of mathematical physics.

Talk 1: Formal Calculus (The Language of VOA Theory)

Content: Formal power series and Laurent series. The formal delta function $\delta(x)$ and its algebraic identities. Formal derivatives and the formal Taylor formula.

Literature: [LL] Sections 2.1 & 2.2.

Talk 2: Axiomatic Definition of Vertex Operator Algebras

Content: Vertex Algebras (VA) as triples $(V, Y, \mathbf{1})$ subject to conditions (truncation, vacuum, creation, Jacobi identity). Transition to Vertex Operator Algebras (VOA) via the conformal vector ω and Virasoro relations.

Literature: [LL] Sections 3.1 (& 3.6 if time permits).

Talk 3: Fundamental Properties and Identities

Content: Deriving key consequences of the Jacobi Identity: Commutativity, Associativity, and the OPE formula.

Literature: [LL] Sections 3.2 & 3.3 (OPE in Remarks 3.3.12 – 3.3.14).

Talk 4: The Virasoro VOA

Content: The Virasoro Lie algebra and construction of the VOA structure on $V_{\mathrm{Vir}}(\ell, 0)$. The modules $M_{\mathrm{Vir}}(\ell, h)$.

Literature: [LL] Section 6.1.

Talk 5: The Heisenberg VOA

Content: Construction of the Heisenberg Lie algebra $\hat{\mathfrak{h}}_*$, the corresponding VOA $V_{\hat{\mathfrak{h}}}(\ell, 0)$ and its modules $M(\ell, \alpha)$. Discussion that the latter are irreducible.

Literature: [LL] Section 6.2 (as a prerequisite) & Section 6.3.

Phase 2: Geometry and the Chiral de Rham Complex

This phase shifts from algebraic foundations to the geometric realization as sheaves on manifolds.

Talk 6: Coordinate Changes and Geometric Vertex Algebras

Content: Construction of the vector bundle $\mathcal{V}$ on a smooth complex curve associated to a quasi-conformal vertex algebra V, and of the flat connection on it.

Literature: [FBZ] Chapter 6. (The main construction is in Section 6.5, transformation of vertex operators under $z \mapsto \rho(z)$ is in Section 6.3 where you also find Definition 6.3.4 of quasi-conformal VAs; explain also that $\mathcal{Y}_x$ is horizontal for the canonical connection.)

Talk 7: Chiral de Rham Complex I: Local Construction

Content: Construct the chiral de Rham complex of $\mathbb{A}^N$.

Literature: [MSV] Sections 1 & 2 (including the proof of Thm. 2.4).

Talk 8: Chiral de Rham Complex II: Globalization and Obstructions

Content: Construct the sheaf of conformal VAs Ω_X^{ch} and discuss the relation to the usual de Rham complex (Thm. 4.4 in the reference).

Literature: [MSV] Sections 3 & 4.

Talk 9: Conformal anomaly

Content: Introduce the gerbe of chiral differential operators and compute the obstruction class; explain the relation to the chiral de Rham complex. Discuss conformal anomaly and the obstruction to the global existence of the Virasoro field.

Literature: Gorbounov–Malikov–Schechtman: *Gerbes of chiral differential operators.*

Talk 10: Elliptic genera and mirror symmetry

Content: Explain that elliptic genera of two mirror symmetric hypersurfaces in toric varieties coincide up to an expected sign.

Literature: Borisov–Libgober: *Elliptic Genera and Applications to Mirror Symmetry* (mainly Section 6 but you have to recall and discuss some stuff from earlier sections like the definition of elliptic genera and that they are weak Jacobi forms; but focus mainly on that parts which use the chiral de Rham complex).

Phase 3: Conformal Blocks

This phase explores the applications of VOAs to conformal field theory.

Talk 11: Conformal Blocks

Content: First define VOA modules. Then give the definition of the space of conformal blocks of a punctured Riemann surface (as the dual of the space of co-invariants). Explain how this yields a vector bundle over the moduli space of punctured Riemann surfaces. Discuss finite-dimensionality of conformal blocks, and the behaviour under insertion of vacua.

Literature: [FBZ] Ch. 9; [LL], Ch. 4; Ueno: *Conformal Field Theory with Gauge Symmetries*, AMS, 2008, Section 3; Zhu: *Modular invariance of characters of a VOA*, JAMS, (1996) Section 4.

Talk 12: Correlation Functions

Content: Definition of the n-point correlation functions $\langle \phi_1(z_1)...\phi_n(z_n)\rangle$, locally, and globally in terms of conformal blocks. Describe the OPEs and fusion rules, and the Ward identities.

Literature: [FBZ] Ch. 4.5; Zhu: *Modular invariance of characters of a VOA*, JAMS, (1996) Section 4; Ueno: *Conformal Field Theory with Gauge Symmetries*, AMS, 2008, Section 3.

Talk 13: The Knizhnik–Zamolodchikov connection

Content: Derivation of the KZ equation using the Ward identities and the Sugawara construction. Interpretation of the KZ equation as a flat connection on the bundle of conformal blocks.

Literature: [FBZ] Ch. 13; Kohno: *Conformal Field Theory and Topology*, Ch. 1.4–1.6.

Talk 14: Monodromy and Braid Group Representations

Content: Discussion of the monodromy of KZ solutions and the induced Braid group actions. Connections to the Jones polynomial and quantum invariants.

Literature: Kohno: *CFT and Topology*, Ch. 2.1–2.2; Kassel: *Quantum Groups*, Ch. 19.

Phase 4: Modularity and Advanced Topics

The final phase connects algebraic structures to arithmetic and modular invariance.

Talk 15: Modular Forms and Partition Functions

Content: Introduce the modular group and its action on the upper half plane, describe a fundamental domain for the action and that the group is generated by two elements (denoted S and T by Serre). Define modular forms, and make your way to the modular function j. Discuss the expansion at infinity of Δ, discuss the Eisenstein series $G_k(z)$ and its q-expantion.

Literature: Serre: *A Course in Arithmetic*, Ch. 7 (§1–§4).

Talk 16: Zhu's Modular Invariance Theorem

Content: Make your way through the paper to discuss at the end modular invariance of rational VOAs (Thm. 5.3.3 in Zhu's paper).

Literature: Zhu: *Modular invariance of characters of VOA*, JAMS (1996).

Talk 17: The Verlinde Formula

Content: Definition of fusion rules and the modular S-matrix. The diagonalization of fusion coefficients via the S-matrix.

Literature: Di Francesco, et al: *Conformal Field Theory*, Ch. 10.8; Verlinde: *Fusion rules and modular transformations in 2D conformal field theory*, Nuclear Physics B, vol. 300 (1988).

Talk 18: Monstrous Moonshine *Content:* Construction of the monster Lie algebra M and proof of the moonshine conjecture for the monster vertex algebra V.

Literature: Gannon: *Moonshine beyond the Monster*; Borcherds: *Monstrous moonshine and monstrous Lie superalgebras*, Invent. math. (1992).