Abstract computation over first-order structures

Sufficient conditions for the existence of universal BSS RAMs

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Outline

From algorithms over first-order structures to universal machines

Motivation: Reasons for our generalization

Introduction: The BSS-RAM model

- Structures $\mathcal{A}$ of signature σ
- σ-Programs
- Transition systems and BSS RAMs
- An example

Semi-decidable decision problems

Universal machines of type 1 and their importance

Constants and their recognizability

Algorithms over First-Order Structures
Informal

Examples

IF $X > 1$ THEN $X := X + 1$.

$\Uparrow\Uparrow$ $\Uparrow\Uparrow$

describes an algorithm to compute $x + 1$ for x greater than 1.

We use

- an operation to transform an object,
- a relation for evaluating a condition,
- a constant.

The underlying structures

of our machines has this form:

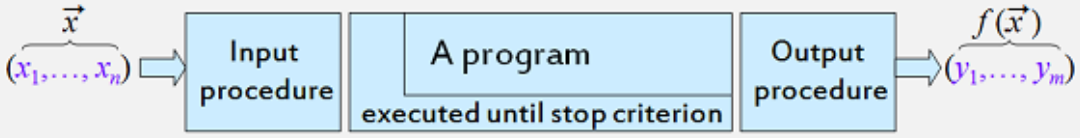
$$\mathcal{A} = (\underbrace{U_{\mathcal{A}}}_{\text{universe}}; \underbrace{(c_i)_{i\in N_1}}_{\text{constants}}; \underbrace{(f_i)_{i\in N_2}}_{\text{operations}}; \underbrace{(r_i)_{i\in N_3}}_{\text{relations}}).$$

Known Machines over First-Order Structures

First-order structures $\mathcal{A}$ and several types of machines

$(\{0,1\};0,1;;=)$	Turing machines (1), Type-2 machines (3)
$(\mathbb{N};0,1;+,-,\cdot;<,=)$	classical RAMs (2)
$(\mathbb{R};\mathbb{R};+,-,\cdot;<,=)$	real RAMs (2), algebraic models (2), BSS machines (1)
$(\mathbb{R};\mathbb{R};+,-;<,=)$	linear models (1), (2), additive BSS machines (1)

Minsky (1961), Scott (1967), Blum/Shub/Smale (1989),... (offline):



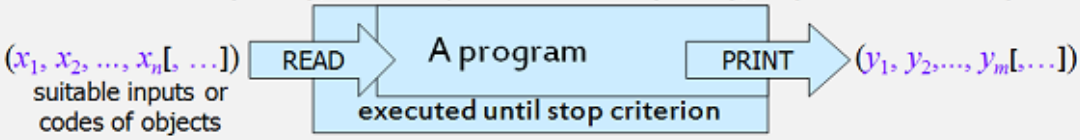
(1)

machines over A

$\mathcal{A}$-machines

BSS RAMs over $\mathcal{A}$

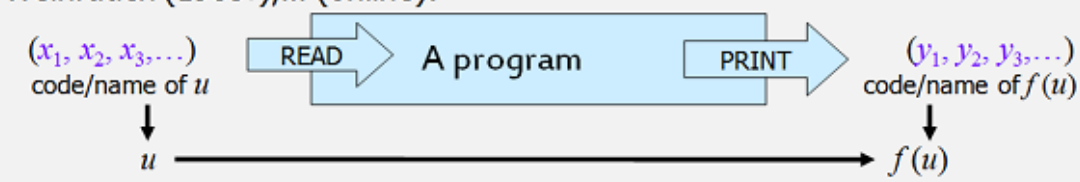
Cook/Reckhow (1973), Aho/Hopcroft/Ullman (1974),... (offline / online):



(2)

machines over A

Weihrauch (1985?),... (online):



(3)

machines over A

Known Machines over First-Order Structures

First-order structures

$(\{0,1\};0,1;;=)$	Turing machines (1)
$(\mathbb{N};0,1;+,-,\cdot;<,=)$	classical RAMs (2)
$(\mathbb{R};\mathbb{R};+,-,\cdot;<,=)$	real RAMs (3), algebraic models (3), BSS machines (4)
$(\mathbb{R};\mathbb{R};+,-;<,=)$	linear models (3), (5), additive BSS machines (5)

Introduced and/or investigated by

(1) Alan M. Turing,

(2) Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman, . . .

(3) Franco P. Preparata, Michael I. Shamos, . . .

(4) Lenore Blum, Steve Smale, Michael Shub,

(5) Felipe Cucker, Klaus Meer, Pascal Koiran, ...

A Program for Semi-Deciding a Set

Over $(\mathbb{R}; \mathbb{Z}; +, \cdot; =)$

Examples (Semi-deciding the square roots)

Input $(x_1, \ldots, x_n) \in \mathbb{R}^\infty$.

1: if $I_1 = I_2$ then goto 2 else goto 1;

2: $Z_2 := -1$;

3: $Z_3 := 1$;

4: $Z_1 := Z_1 * Z_1$;

5: $Z_2 := Z_2 + Z_3$;

6: if $Z_1 = Z_2$ then goto 7 else goto 5;

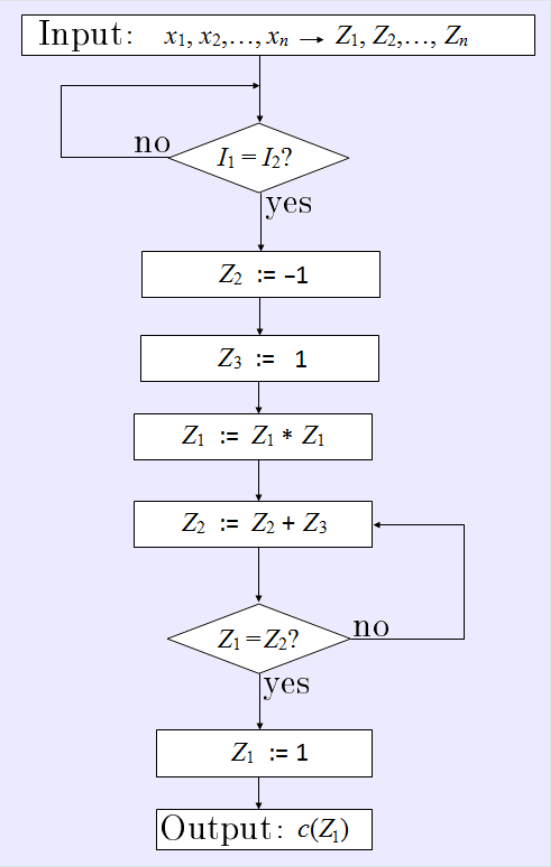
7: $Z_1 := 1$;

8: stop.

Output $c(Z_1)$.

Symbols: $+$ and $*$ for binary functions.
1 and -1 for a constant.

Infix notation $Z_1 + Z_2$ stands for $f_1^2(Z_1, Z_2)$ and so on.

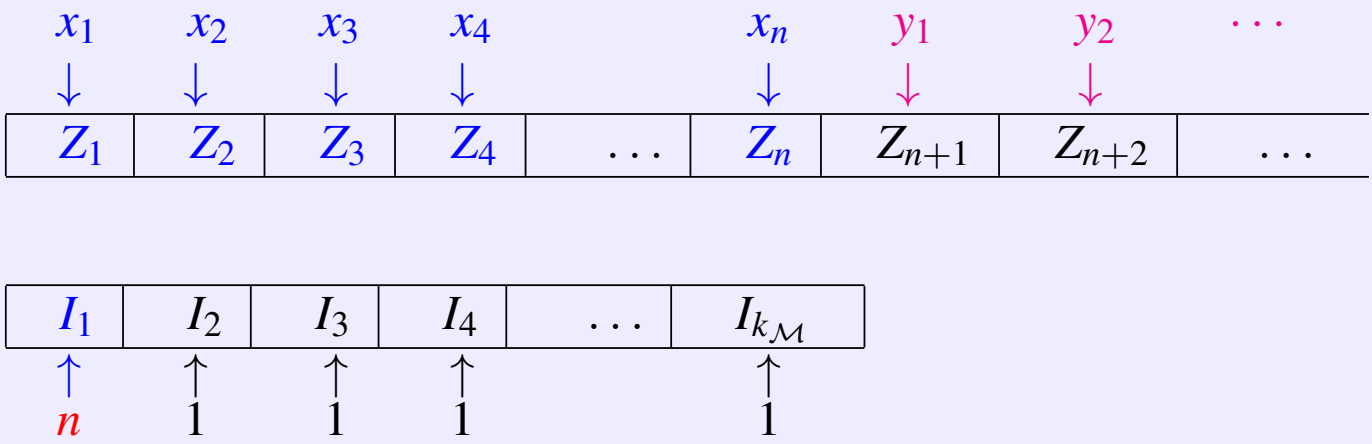


Uniform Computation over First-Order Structures

Input and output procedures for [digitally] non-deterministic BSS RAMs in $\mathrm{M}_{\mathcal{A}}^{[\mathrm{D}]\,\mathrm{ND}}$

Input and output space: $U_{\mathcal{A}}^{\infty} =_{\mathrm{df}} \bigcup_{i \geq 1} U_{\mathcal{A}}^{i}$

Input of $\vec{x} = (x_1, \ldots, x_n) \in U_{\mathcal{A}}^{\infty}$:



with guesses $y_1, \ldots, y_m \in U_{\mathcal{A}}$ or
"digits" $y_1, \ldots, y_m \in \{c_1, c_2\}$

Output after a stop-instruction:

$(c(Z_1), \ldots, c(Z_{c(I_1)}))$

Algorithms over First-Order Structures

σ-programs for [in]finite $\mathcal{A}$-machines $\mathcal{M}$

A σ-program can be interpreted by any structure of a finite or infinite suitable signature.

To simplify matters, let us consider

- a structure $\mathcal{A} = (U_{\mathcal{A}}; c_1, \ldots, c_{n_1}; f_1, \ldots, f_{n_2}; r_1, \ldots, r_{n_3})$
- a finite signature $\sigma = (n_1; m_1, \ldots, m_{n_2}; k_1, \ldots, k_{n_3})$
- the first-order symbols $c_1^0, \ldots, c_{n_1}^0, f_1^{m_1}, \ldots, f_{n_2}^{m_{n_2}}, r_1^{k_1}, \ldots, r_{n_3}^{k_{n_3}}$

Definition (σ-program $\mathcal{P}_{\mathcal{M}}$ of an $\mathcal{A}$-machine)

$1 :$ instruction$_1$; $\ldots$; $\ell_{\mathcal{P}_{\mathcal{M}}} - 1 :$ instruction$_{\ell_{\mathcal{P}_{\mathcal{M}}}-1}$; $\ell_{\mathcal{P}_{\mathcal{M}}} :$ stop.

- Any instruction$_i$ is a σ-instruction.
- The execution of $\mathcal{P}_{\mathcal{M}}$ is:
  the stepwise transformation of configurations
  by the transition system $(\mathrm{S}_{\mathcal{M}}, \rightarrow_{\mathcal{M}})$.

Algorithms over First-Order Structures

σ-instructions for infinite $\mathcal{A}$-machines $\mathcal{M}$ and transport instructions

Z_1	Z_2	Z_3	Z_4	Z_5	$\ldots$

Z-registers (for *individuals* in $U_{\mathcal{A}}$)

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_{\mathcal{M}}}$

Index registers (for *indices* in $\mathbb{N}_+$)

Computation	$\ell\colon\ Z_j := f_i^{m_i}(Z_{j_1}, \ldots, Z_{j_{m_i}})$	(1)
	$\ell\colon\ Z_j := c_i^0$	(2)
Copy	$\ell\colon\ Z_{I_j} := Z_{I_k}$	(3)
Branching	$\ell\colon$ if $r_i^{k_i}(Z_{j_1}, \ldots, Z_{j_{k_i}})$ then goto ℓ_1 else goto ℓ_2	(4)
Index	$\ell\colon\ I_j := 1$	(5)
	$\ell\colon\ I_j := I_j + 1$	(6)
	$\ell\colon$ if $I_j = I_k$ then goto ℓ_1 else goto ℓ_2	(7)
Stop	$l\colon$ stop	(8)

The Z-registers contain individuals of the underlying structure. For copying these values from one Z-register into another Z-register, we will use index registers and index instructions.

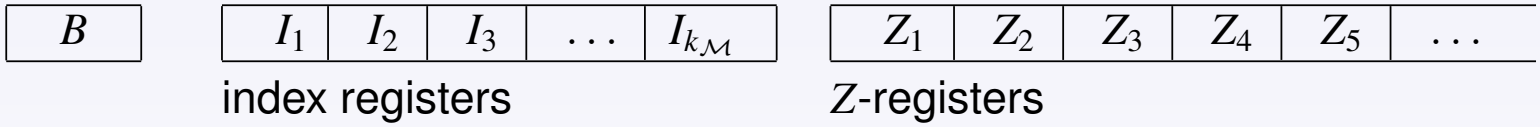
Overall States

Configurations of an $\mathcal{A}$-machine $\mathcal{M}$

Definition (Configurations in $S_{\mathcal{M}}$)

$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \quad = \underbrace{(\ell, \nu_1, \ldots, \nu_{k_{\mathcal{M}}}, u_1, u_2, \ldots)}$$

a configuration of $\mathcal{M}$



index registers

Z-registers

pointers

read-and-write heads

tape

$\ell \in \mathcal{L}_{\mathcal{M}}$ $\quad \vec{\nu} = (\nu_1, \ldots, \nu_{k_{\mathcal{M}}}) \in \mathbb{N}_+^{k_{\mathcal{M}}}$ $\quad \bar{u} = (u_1, u_2, \ldots) \in U_{\mathcal{A}}^{\omega}$

- ℓ a label in $\mathcal{L}_{\mathcal{M}}$ — instruction counter, the internal state
- $\vec{\nu}$ indices of Z-registers — addresses, positions on the tape
- $\bar{u}$ a sequence of individuals

Unit Costs by a Computational System for $\mathcal{A}$-Machines

Transformation of configurations of an $\mathcal{A}$-machine $\mathcal{M}$

Definition (Transition system for any $\mathcal{A}$-machine $\mathcal{M}$)

$$\mathcal{S}_{\mathcal{M}} = (\mathsf{S}_{\mathcal{M}}, \rightarrow_{\mathcal{M}})$$

with a binary relation $\rightarrow_{\mathcal{M}} \subseteq \mathsf{S}^2_{\mathcal{M}}$

defined below.

$\rightarrow_{\mathcal{M}}$ depends on

- the types of the single instructions belonging to the program,
- the single parameters in these instructions, and
- the underlying structure.

(cf. Introduction 2020, . . .; see also Börger for finite machines, . . .)

Unit Costs of $\mathcal{A}$-machines

Transformation of configurations of an $\mathcal{A}$-machine $\mathcal{M}$

$\ell\colon\ Z_j := f_i^{m_i}(Z_{j_1}, \ldots, Z_{j_{m_i}})$

$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \quad \to_{\mathcal{M}} \quad (\ell + 1 \,.\, \vec{\nu} \,.\, (u_1, \ldots, u_{j-1}, f_i(u_{j_1}, \ldots, u_{j_{m_i}}), u_{j+1}, \ldots))$$

$\ell\colon\ Z_j := c_i^0$

$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \quad \to_{\mathcal{M}} \quad (\ell + 1 \,.\, \vec{\nu} \,.\, (u_1, \ldots, u_{j-1}, c_i, u_{j+1}, \ldots))$$

$\ell\colon\ Z_{I_j} := Z_{I_k}$

$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \quad \to_{\mathcal{M}} \quad (\ell + 1 \,.\, \vec{\nu} \,.\, (u_1, \ldots, u_{\nu_j - 1}, u_{\nu_k}, u_{\nu_j + 1}, \ldots))$$

$\ell\colon$ if $r_i^{k_i}(Z_{j_1}, \ldots, Z_{j_{k_i}})$ then goto ℓ_1 else goto ℓ_2

$$\begin{array}{lll} (\ell \,.\, \vec{\nu} \,.\, \bar{u}) & \to_{\mathcal{M}} \quad (\ell_1 \,.\, \vec{\nu} \,.\, \bar{u}) & \text{if } (u_{j_1}, \ldots, u_{j_{k_i}}) \in r_i \\ (\ell \,.\, \vec{\nu} \,.\, \bar{u}) & \to_{\mathcal{M}} \quad (\ell_2 \,.\, \vec{\nu} \,.\, \bar{u}) & \text{if } (u_{j_1}, \ldots, u_{j_{k_i}}) \notin r_i \end{array}$$

Unit Costs of $\mathcal{A}$-machines

Transformation of configurations of an $\mathcal{A}$-machine $\mathcal{M}$

ℓ: if $I_j = I_k$ then goto ℓ_1 else goto ℓ_2

$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \;\to_{\mathcal{M}}\; (\ell_1 \,.\, \vec{\nu} \,.\, \bar{u}) \qquad \text{if } \nu_j = \nu_k$$
$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \;\to_{\mathcal{M}}\; (\ell_2 \,.\, \vec{\nu} \,.\, \bar{u}) \qquad \text{if } \nu_j \neq \nu_k$$

ℓ: $I_j := 1$

$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \;\to_{\mathcal{M}}\; (\ell + 1 \,.\, (\nu_1, \ldots, \nu_{j-1}, 1, \nu_{j+1}, \ldots) \,.\, \bar{u})$$

ℓ: $I_j := I_j + 1$

$$(\ell \,.\, \vec{\nu} \,.\, \bar{u}) \;\to_{\mathcal{M}}\; (\ell + 1 \,.\, (\nu_1, \ldots, \nu_{j-1}, \nu_j + 1, \nu_{j+1}, \ldots) \,.\, \bar{u})$$

$\ell_{\mathcal{P}}$: stop

$$(\ell_{\mathcal{P}} \,.\, \vec{\nu} \,.\, \bar{u}) \;\to_{\mathcal{M}}\; (\ell_{\mathcal{P}} \,.\, \vec{\nu} \,.\, \bar{u})$$

Semi-Decidable Problems: Halting Sets

Some decision problems of BSS RAMs

Definition (Decision problems)

Any subset P of $U_{\mathcal{A}}^{\infty}$ (that contains all tuples) is a *decision problem*.

Definition (Halting process)

$\mathcal{M}$ *halts* on input $\vec{x}$ if $\ell_{\mathcal{P}}$: stop is reached [for some guesses].

$\mathcal{M}(\vec{x})\downarrow$ if $\mathcal{M}$ halts on input $\vec{x}$ after a finite number of steps [for some guesses].

$\mathcal{M}(\vec{x})\uparrow$ if $\mathcal{M}(\vec{x})\downarrow$ does not hold.

Definition (Halting set)

$H_{\mathcal{M}} = \{\vec{x} \in U_{\mathcal{A}}^{\infty} \mid \mathcal{M}(\vec{x})\downarrow\}$ is the *halting set* of $\mathcal{M}$.

$\vec{x} \in H_{\mathcal{M}} \Rightarrow$ $\mathcal{M}$ *accepts* $\vec{x}$.

$\mathrm{Res}_{\mathcal{M}}(\vec{x}) \in U_{\mathcal{A}}^{\infty}$ and $\emptyset \neq \mathrm{Res}_{\mathcal{M}}(\vec{x}) \subseteq U_{\mathcal{A}}^{\infty}$ (resp.)

$\mathrm{SDEC}_{\mathcal{A}} = \{H_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\}$ $\quad$ $\mathrm{SDEC}_{\mathcal{A}}^{\mathrm{ND}} = \{H_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}^{\mathrm{ND}}\}$

Complete Problems: Halting Problems

Over first-order structures

Properties (Halting problems)

$Halt = \{\langle input, machine\rangle \subseteq U_{\mathcal{A}}^{\infty} \mid \langle machine\rangle \text{ halts on } \langle input\rangle\}$

$\Rightarrow$ It is useful to consider universal BSS RAMs.

Universal Machines of Type 1 and their Halting Sets

Halting problems

$$\mathcal{A} = (U_{\mathcal{A}}; (c_i)_{i \in N_1}; f_1, \ldots, f_{n_2}; r_1, \ldots, r_{n_3}) \qquad (\{1,2\} \subseteq N_1)$$

Type 1 here means encoding by using two constants

$$a = c_1,$$
$$b = c_2.$$

$$\mathrm{code}_{(a,b)}(\mathcal{M}) \in \{a,b\}^\infty \qquad \text{(for all BSS RAMs } \mathcal{M} \text{ over } \mathcal{A})$$

Definition (Halting problems of type 1)

$$\mathbb{H}_{\mathcal{A}} = \{(\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x}) \in U_{\mathcal{A}}^\infty \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}} \quad \& \; \mathcal{M}(\vec{x}) \downarrow\}$$

$$\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}} = \{(\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x}) \in U_{\mathcal{A}}^\infty \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}^{\mathrm{ND}} \quad \& \; \mathcal{M}(\vec{x}) \downarrow\}$$

for BSS RAMs over $\mathcal{A}$, in $\mathsf{M}_{\mathcal{A}}^{[\mathrm{ND}]}$, without guessing, with guessing

Simulation of $\mathcal{M}$ by a Universal Machine $\mathcal{M}_0$ (Type 1)

Computing $\mathrm{Res}_{\mathcal{M}} = \mathrm{Res}_{\mathcal{M}_0} \circ \mathrm{Res}_{\mathcal{N}_{\mathcal{M}}}$ for $\mathcal{A}$ with $\{1,2\} \subseteq N_1$

$\boxed{\vec{x}}$ $\Rightarrow_{\mathcal{N}_{\mathcal{M}}}$ $\boxed{(\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x})}$

$\Downarrow \mathrm{Input}_{\mathcal{M}}$ $\Downarrow \mathrm{Input}_{\mathcal{M}_0}$

initial configuration for $\mathcal{M}$

$\boxed{(1 \,.\, (n, 1, \ldots, 1) \,.\, \vec{x} \,.\, (x_n, x_n, \ldots))}$

initial configuration for $\mathcal{M}_0$

$\boxed{(1 \,.\, (n_0, 1, \ldots, 1) \,.\, (\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x} \,.\, (x_n, x_n, \ldots)))}$

with $n_0 = |(\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x})|$

$\Downarrow$ The execution of $\mathcal{M}$ $\Downarrow$ The execution of $\mathcal{M}_0$

end configuration

$\boxed{(\ell_{\mathcal{P}_{\mathcal{M}}} \,.\, \vec{\nu} \,.\, \bar{u})}$

end configuration

$\boxed{(\ell_{\mathcal{P}_{\mathcal{M}_0}} \,.\, \vec{\mu} \,.\, \bar{z})}$

$\Downarrow \mathrm{Output}_{\mathcal{M}}$ $\Downarrow \mathrm{Output}_{\mathcal{M}_0}$

$\boxed{\mathrm{Res}_{\mathcal{M}}(\vec{x}) = (u_1, \ldots, u_{\nu_1})}$ $=$ $\boxed{(z_1, \ldots, z_{\mu_1}) = \mathrm{Res}_{\mathcal{M}_0}(\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x})}$

Simulation of $\mathcal{M}$ by a Universal Machine $\mathcal{M}_0$ (Type 1)

The reduction of halting sets

Input of $\mathcal{M}$		Input of $\mathcal{M}_0$
$\vec{x}$	$\Rightarrow_{\mathcal{N}_\mathcal{M}}$	$(\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x})$

$\Rightarrow$ $\mathcal{N}_\mathcal{M}$ computes the reduction of $H_\mathcal{M}$ to $H_{\mathcal{M}_0}$

for $\mathcal{M} \in \mathsf{M}_\mathcal{A}$ (in linear time).

$$
\begin{array}{rcl}
\vec{x} \in H_\mathcal{M} & \Rightarrow & (\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x}) \in H_{\mathcal{M}_0} \\
\vec{x} \notin H_\mathcal{M} & \Rightarrow & (\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x}) \notin H_{\mathcal{M}_0} \\
\mathcal{M} \text{ halts on } \vec{x} & \Leftrightarrow & \mathcal{M}_0 \text{ halts on } (\mathrm{code}_{(a,b)}(\mathcal{M}) \,.\, \vec{x})
\end{array}
$$

Note, we have also:

$\mathcal{N}_\mathcal{M}$ computes the reduction of $H_\mathcal{M}$ to $H_{\mathcal{M}_0^{\mathrm{ND}}}$ for $\mathcal{M} \in \mathsf{M}_\mathcal{A}^{\mathrm{ND}}$ (in linear time).

Encoding BSS RAMs $\mathcal{M}$ for their Simulation

Strings in $\{a,b\}^\kappa$ as codes for the σ-programs

Our instructions $\ell\colon\ Z_j := f_i^{m_i}(Z_{j_1},\ldots,Z_{j_{m_i}})$ (1), ... encoded by

$code_\lambda(\cdot)$ in $\{a,b\}^\lambda$ (codes for labels and indices)

$\Rightarrow (code_\lambda(\cdot)\,.\ code_\lambda(\cdot)\,.\ldots)$ in $\{a,b\}^\kappa$ (codes for instructions)

Type	e_1	e_2	e_3	e_4	e_5	e_6	e_7	$\cdots$	$e_{\ldots}$	Codes of length κ
(1)	1	i			j	j_1	j_2		j_{m_i}	$(code_\lambda(1)\,.\ code_\lambda(i)\,.\ldots)$
(2)	2				j					$(code_\lambda(2)\,.\ code_\lambda(0)\,.\ldots)$
(3)	3				j	k				
(4)	4	i	ℓ_1	ℓ_2		j_1	j_2		j_{k_i}	
(5)	5		ℓ_1	ℓ_2	j	k				
(6)	6				j					
(7)	7				j					
(8)	0									

Recall: $a = c_1$, $b = c_2$.

Are the codes (are c_1 and c_2) recognizable or distinguishable?

Subprograms for Evaluating Codes?

Subprograms for deciding codes?

The equality relation = can be used in executing a branching instruction of type (4) only if the relation = belongs to the underlying structure $\mathcal{A}$.

- A structure $\mathcal{A} = (U_\mathcal{A}; (c_i)_{i\in N_1}; f_1, \ldots, f_{n_2}; r_1, \ldots, r_{n_3})$ $\quad (\{1,2\} \subseteq N_1)$
- an expansion $\quad \mathcal{A}^+ = (U_\mathcal{A}; (c_i)_{i\in N_1}; f_1, \ldots, f_{n_2}; r_1, \ldots, r_{n_3}, =)$
- a new signature $\quad \sigma^+ = (n_1; m_1, \ldots, m_{n_2}; k_1, \ldots, k_{n_3}, 2)$
- new symbols (for the identity) $\quad =, \mathrm{id}, r^2_{n_3+1}$

Examples (Useful pseudo instructions for computation over $\mathcal{A}$?)

ℓ: if $\mathrm{id}(\mathrm{Z}_1, \mathrm{c}_1^0)$ then goto ℓ_1 else goto ℓ_2 (i)

ℓ: if $\mathrm{id}(\mathrm{Z}_1, \mathrm{c}_2^0)$ then goto ℓ_1 else goto ℓ_2 (ii)

id is a symbol for a binary relation. id could be interpreted, e.g., by the equality relation =.

If the equality relation is decidable over $\mathcal{A}$ by a subprogram, then both pseudo instructions can stand for subprograms allowing to decide $\{c_1\}$ and $\{c_2\}$.

Which properties must $\mathcal{A}$ possess in order to be able to evaluate the codes?

Subprograms for Evaluating Codes

Sufficient conditions for subprograms helping to decide codes

Definition (The classes $\mathrm{SDEC}_{\mathcal{A}}$ and $\mathrm{DEC}_{\mathcal{A}}$)

$$\mathrm{SDEC}_{\mathcal{A}} = \{H_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\}$$
$$\mathrm{DEC}_{\mathcal{A}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \in \mathrm{SDEC}_{\mathcal{A}} \ \& \ U_{\mathcal{A}}^{\infty} \setminus P \in \mathrm{SDEC}_{\mathcal{A}}\}$$

(a) $\mathcal{A}$ contains $\mathrm{id}_{U_{\mathcal{A}}}$,
(b) $\mathrm{id}_{U_{\mathcal{A}}} \in \mathrm{DEC}_{\mathcal{A}}$,
(c) $\mathrm{id}_{U_{\mathcal{A}}} \in \mathrm{SDEC}_{\mathcal{A}}$,
(d) $\{c_1\}, \{c_2\} \in \mathrm{SDEC}_{\mathcal{A}}$.

Theorem (Pseudo instructions for evaluating codes over $\mathcal{A}$)

Let one of conditions, (a) or (b) or (c) or (d), hold.

Then, $\{c_1\}, \{c_2\} \in \mathrm{DEC}_{\mathcal{A}}$ and

(i) and (ii) describe subprograms for deciding $\{c_1\}$ and $\{c_2\}$ over $\mathcal{A}$.

ℓ: if $\mathrm{id}(Z_1, c_1^0)$ then goto ℓ_1 else goto ℓ_2 (i)

ℓ: if $\mathrm{id}(Z_1, c_2^0)$ then goto ℓ_1 else goto ℓ_2 (ii)

More Background (for Evaluating Codes)
Identity, constants, and basic implications

Definition (The classes $\mathrm{SDEC}_{\mathcal{A}}$ and $\mathrm{DEC}_{\mathcal{A}}$)

$$\begin{aligned}\mathrm{SDEC}_{\mathcal{A}} &= \{H_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\}\\ \mathrm{DEC}_{\mathcal{A}} &= \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \in \mathrm{SDEC}_{\mathcal{A}} \;\&\; U_{\mathcal{A}}^{\infty} \setminus P \in \mathrm{SDEC}_{\mathcal{A}}\}\end{aligned}$$

$$\begin{array}{lll}(a) & \mathcal{A} \text{ contains } & \mathrm{id}_{U_{\mathcal{A}}},\\ (b) & \mathrm{id}_{U_{\mathcal{A}}} & \in \mathrm{DEC}_{\mathcal{A}},\\ (c) & \mathrm{id}_{U_{\mathcal{A}}} & \in \mathrm{SDEC}_{\mathcal{A}},\\ (d) & \{c_1\}, \{c_2\} & \in \mathrm{SDEC}_{\mathcal{A}},\\ (e) & \{c_1, c_2\}^{\infty} & \in \mathrm{SDEC}_{\mathcal{A}}.\end{array}$$

Let $(a) \Rightarrow (b)$ mean (a) implies (b), ...

There hold

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \qquad \text{(cf. Part IIb)}$$

and there are structures with

$$(a) \not\Leftarrow (b) \quad \text{or} \quad (c) \not\Leftarrow (d) \quad \text{or} \quad (d) \not\Leftarrow (e). \qquad \text{(cf. Part IIb)}$$

What do we have, (b) $\not\Leftarrow$ (c) or (b) $\Leftarrow$ (c) ?

More Background (for Evaluating Codes)

Basic properties and some equivalences

Definition (The classes $\mathrm{SDEC}_{\mathcal{A}}$ and $\mathrm{DEC}_{\mathcal{A}}$)

$$\mathrm{SDEC}_{\mathcal{A}} = \{H_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\}$$
$$\mathrm{DEC}_{\mathcal{A}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \in \mathrm{SDEC}_{\mathcal{A}} \;\&\; U_{\mathcal{A}}^{\infty} \setminus P \in \mathrm{SDEC}_{\mathcal{A}}\}$$

$$\begin{array}{lll} (a) & \mathcal{A} \text{ contains} & \mathrm{id}_{U_{\mathcal{A}}}, \\ (b) & \mathrm{id}_{U_{\mathcal{A}}} & \in \mathrm{DEC}_{\mathcal{A}}, \\ (c) & \mathrm{id}_{U_{\mathcal{A}}} & \in \mathrm{SDEC}_{\mathcal{A}}, \\ (d) & \{c_1\}, \{c_2\} & \in \mathrm{SDEC}_{\mathcal{A}}, \\ (e) & \{c_1, c_2\}^{\infty} & \in \mathrm{SDEC}_{\mathcal{A}}. \end{array}$$

Let $(a) \Leftrightarrow (b)$ mean $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$, ...

Then:

$$\begin{array}{llll} (c) & \Leftrightarrow & \mathrm{id}_{U_{\mathcal{A}}} & \in \mathrm{DEC}_{\mathcal{A}}. \\ (d) & \Leftrightarrow & \{c_1\}, \{c_2\} & \in \mathrm{DEC}_{\mathcal{A}}. \\ (e) & \Leftrightarrow & \{c_1, c_2\} & \in \mathrm{DEC}_{\mathcal{A}}. \end{array}$$

(cf. Part IIb) (cf. Part IIb) (cf. Part IIb)

Thus:

$$\boxed{(b) \Leftrightarrow (c)}$$
$$\Leftrightarrow \quad \chi_{\mathrm{id}_{U_{\mathcal{A}}}} \in \{\mathrm{Res}_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\}.$$

(see below)

More: Decidability of P and Computability of χ_P

Over structures with two constants and $P \subseteq U_{\mathcal{A}}^{\infty}$

Definition (The characteristic function)

$$\chi_P : U_{\mathcal{A}}^{\infty} \to \{c_1, c_2\} \quad \textit{and} \quad \chi_P(\vec{x}) = \begin{cases} c_1 & \textit{if } \vec{x} \in P, \\ c_2 & \textit{if } \vec{x} \in U_{\mathcal{A}}^{\infty} \setminus P \end{cases}.$$

$$P \in \mathrm{DEC}_{\mathcal{A}} \quad \Rightarrow \quad \chi_P \in \{\mathrm{Res}_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\}. \qquad \text{(cf. Part IIb)}$$

The computation of χ_P by an $\mathcal{M} \in \mathsf{M}_{\mathcal{A}}$ should mean that

- $\chi_P(\vec{x}) = c_1$ means $\vec{x} \in P$ and $\vec{x}$ is accepted by $\mathcal{M}$,
- $\chi_P(\vec{x}) = c_2$ means $\vec{x} \in U_{\mathcal{A}}^{\infty} \setminus P$ and $\vec{x}$ is rejected by $\mathcal{M}$.

More: Decidability of P and Computability of χ_P

Over structures with two constants and $P \subseteq U_{\mathcal{A}}^{\infty}$

Definition (The characteristic function)

$$\chi_P : U_{\mathcal{A}}^{\infty} \to \{c_1, c_2\} \quad \textit{and} \quad \chi_P(\vec{x}) = \begin{cases} c_1 & \textit{if } \vec{x} \in P, \\ c_2 & \textit{if } \vec{x} \in U_{\mathcal{A}}^{\infty} \setminus P \end{cases}.$$

$$P \in \mathrm{DEC}_{\mathcal{A}} \quad \Rightarrow \quad \chi_P \in \{\mathrm{Res}_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\}. \qquad \text{(cf. Part IIb)}$$

What about with "$\Leftarrow$"?

$$\chi_P \in \{\mathrm{Res}_{\mathcal{M}} \mid \mathcal{M} \in \mathsf{M}_{\mathcal{A}}\} \text{ and } (d) \quad \Rightarrow \quad P \in \mathrm{DEC}_{\mathcal{A}}.$$

(for several proofs see Part IIb)

Recall: $(a) \Rightarrow (b) \Leftrightarrow (c) \Rightarrow (d)$

where

(a)	$\mathcal{A}$ contains	$\mathrm{id}_{U_{\mathcal{A}}}$,
(b)	$\mathrm{id}_{U_{\mathcal{A}}}$	$\in \mathrm{DEC}_{\mathcal{A}}$,
(c)	$\mathrm{id}_{U_{\mathcal{A}}}$	$\in \mathrm{SDEC}_{\mathcal{A}}$,
(d)	$\{c_1\}, \{c_2\}$	$\in \mathrm{SDEC}_{\mathcal{A}}$.
(e)	$\{c_1, c_2\}$	$\in \mathrm{SDEC}_{\mathcal{A}}$ is not sufficient.

Thanks

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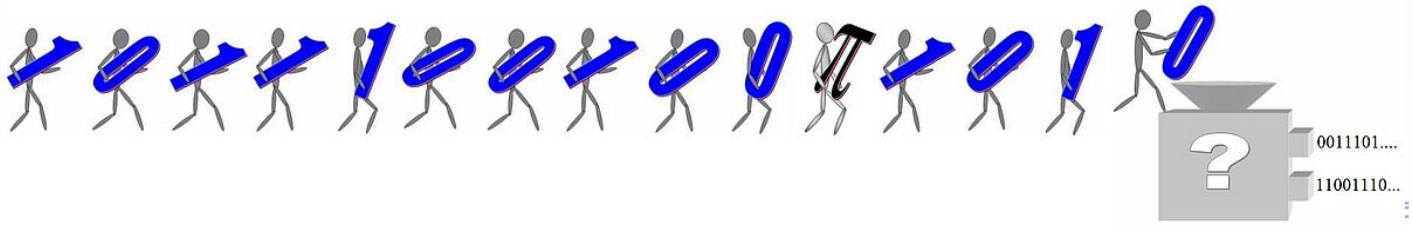

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