RETREAT AT KLOSTER, HIDDENSEE:

OPEN PROBLEMS

AUGUST 2016

These questions were discussed at the retreat at Kloster, Hiddensee.
See https://math-inf.uni-greifswald.de/institut/ueber-uns/mitarbeiter/gassner/arbeitstreffen2016/.

1. Vasco Brattka

The uniform computational power of Martin-Löf randomness can be characterized in the Weihrauch lattice by $\mathsf{MLR} \equiv_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$ (B. and Pauly 2016). Similar characterizations of Peano arithmetic and cohesivness yield $\mathsf{PA} \equiv_{\mathrm{W}} (\mathsf{C}'_{\mathbb{N}} \to \mathsf{WKL})$ and $\mathsf{COH} \equiv_{\mathrm{W}} (\lim \to \mathsf{WKL}')$ (B., Hendtlass and Kreuzer 2016).

Question 1.1. Are there any such characterizations of other randomness notions as implications in the Weihrauch lattice?

We can characterize non-deterministically computable functions f by $f \leq_{\mathrm{W}} \mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}}$ (B., de Brecht, Pauly 2012) and Las Vegas computable functions by $f \leq_{\mathrm{W}} \mathsf{WWKL} \equiv_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}}$ (B., Gherardi, Hölzl 2015).

Question 1.2. Is there a meaningful notion of quantum computability and possibly a suitable choice problem C such that f is quantum computable if and only if $f \leq_{\mathrm{W}} C$ holds?

2. Rupert Hölzl

Two open questions relating to my talk:

(1) We know the following from recent paper with Porter (Randomness for computable measures and initial segment complexity, APAL, to appear).

Theorem 2.1. *Let $X \in 2^{\omega}$ be non-computable, anti-complex, and proper. Then X is high.*

To be *proper* means: ML-random for some computable measure.
To be *anticomplex* means: for every computable order h, we have

$$\forall n\, \mathrm{K}(X \upharpoonright h(n)) \leq^{+} n.$$

We also know the following partial converse.

Theorem 2.2. *Suppose that $X \in \mathsf{MLR}$ and $f \leq_{\mathrm{wtt}} X$ dominates all computable functions. Then there is an anti-complex proper sequence $Y \equiv_{\mathrm{T}} X$.*

Do we get the full converse?

Question 2.3. If $X \in \mathsf{MLR}$ is high, is there some anti-complex proper $Y \equiv_{\mathrm{T}} X$?

(2) A computable measure μ is diminutive if the complement of every component of μ's universal ML-test does not contain a computably perfect subset. By the results in my talk, if X is random w.r.t. to some computable diminutive measure, it cannot be random w.r.t. any computable continuous measure.

Question 2.4. Let X be proper and assume that X is *not* random w.r.t. *any* computable diminutive measure. Is X then random w.r.t. to some computable continuous measure, and therefore complex?

3. André Nies

Question 3.1. Classify the statements "every nondecreasing function on $[0, 1]$ is somewhere [almost everywhere] differentiable" in the Weihrauch lattice.

- Is one of them equivalent to computable randomness `CR`? (This has been done in reverse maths in work with Yokoyama. Should be easier here as one doesn't have to worry about levels of induction.)
- Given that the classical proof is via Vitali coverings, how is the strength of this statement connected to the various versions of VTC in the lattice?

Comment (Arno): If "Every nondecreasing function on $[0, 1]$ is somewhere differentiable" just asks for some x such that $f'(x)$ exists, it is computable. So is the almost everywhere version, if it asks for a Π^0_3-set of measure 1 where the function is differentiable. Maybe we should ask for both x and $f'(x)$?

A sequence of n qbits is given by a unit vector in $\mathcal{H}^{\otimes n}$ where $\mathcal{H} = \mathbb{C}^2$.

Question 3.2 (With Volkher Scholz)**.** Find the right mathematical setting for infinite sequences of qbits.

We say that $Z \in 2^{\mathbb{N}}$ is density random if Z is ML-random and every Π^0_1 class $\mathcal{P}$ with $\mathcal{P} \ni Z$ has Lebesgue density 1 at Z. This is the same as: every left-r.e. Martingale converges along Z. Z is Oberwolfach random if it passes each left-r.e. test, i.e. (G_m) uniformly Σ^0_1 with $\lambda G_m \leq \beta - \beta_m$ for left-r.e. real β. Same as ML-random and computes no K-trivial.

Question 3.3. Is density random the same as Oberwolfach random?

4. Arno Pauly

Question 4.1. Is SR $\equiv_{\mathrm{W}}$ CR? If not, what can we say about SR $\to$ CR? Should CR $\leq_{\mathrm{W}}$ $\mathrm{C}_{\mathbb{N}} \star$ SR hold?

The following questions might be useful as stepping stones towards a resolution of Question 1.1:

Question 4.2. Can we characterize $(\mathsf{C}'_{\mathbb{N}} \to \mathsf{WWKL})$? We know that this principle lies below PA and MLR.

Question 4.3. We have $\mathsf{C}_{\mathbb{N}} \to (\mathsf{C}_{\mathbb{N}} \star \mathsf{S}R) \equiv_{\mathrm{W}} \mathsf{S}R$. Thus, if $(\mathsf{C}_{\mathbb{N}} \star \mathsf{S}R)$ is equivalent to some *nice* problem, this would be an answer to Question 1.1. The same holds for $\mathsf{C}R$ in place of $\mathsf{S}R$.

5. Philipp Schlicht

Question 5.1. (Brattka) Is the an extension of $L_\lambda \prec_{\Sigma_1} L_\zeta \prec_{\Sigma_2} L_\Sigma$ to a fourth class, in other words, is there some $\theta > \Sigma$ such that $L_\Sigma \prec_{\Sigma_3} L_\theta$?

Here, λ, ζ and Σ are the suprema of ITTM-writable, ITTM-eventually writable and ITTM-accidentally writable ordinals, and the lexicographically least triple with $L_\lambda \prec_{\Sigma_1} L_\zeta \prec_{\Sigma_2} L_\Sigma$.

Answer: No. Suppose that $\alpha < \beta < \gamma < \delta$ and $L_\alpha \prec_{\Sigma_1} L_\beta \prec_{\Sigma_2} L_\gamma \prec_{\Sigma_3} L_\delta$. Since $L_\gamma \prec_{\Sigma_3} L_\delta$, there are $\alpha' < \beta' < \gamma' < \gamma'$ with $L'_\alpha \prec_{\Sigma_1} L'_\beta \prec_{\Sigma_2} L'_\gamma$.

However, we can consider the lexicographically least $\alpha < \beta < \gamma < \delta$ with $L_\alpha \prec_{\Sigma_1} L_\beta \prec_{\Sigma_2} L_\gamma \prec_{\Sigma_3} L_\delta$ and the corresponding classes of 'computable' functions, and similarly for arbitrary n. This should allow analogues of Sacks' theorem.

Is $\Sigma < \alpha$ for α as in the previous paragraph?

Question 5.2. (related to a project with Andre) Suppose that K is a compact metric space. Is the set of Polish spaces with $d_{\mathrm{GH}}(K, X)$ Borel? Here d_{GH} denotes the Gromov-Hausdorff distance.

Question 5.3. Is there a characterization of Π^1_1-ML-randomness in the Weihrauch lattice, similar to the result $\mathsf{MLR} \equiv_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$ (Brattka and Pauly 2016)?