Probabilistic Computability and Randomness in the Weihrauch Lattice

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based on different joint work with

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Outline

Algorithmic Randomness in the Weihrauch Lattice

Mathematical Problems and Solutions

Definition

A problem is a partial multi-valued function $f :\subseteq X \rightrightarrows Y$ on represented spaces X, Y.

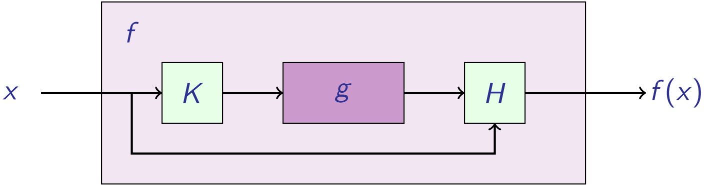
- There are a certain sets of potential inputs X and outputs Y.
- $D = \mathrm{dom}(f)$ contains the valid instances of the problem.
- $f(x)$ is the set of solutions of the problem f for instance x.

Definition

$g :\subseteq X \rightrightarrows Y$ solves $f :\subseteq X \rightrightarrows Y$, if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g(x) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$. We write $g \sqsubseteq f$ in this situation.

Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be two mathematical problems.



- f is Weihrauch reducible to g, $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \times W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $H(\mathrm{id}_X, gK) \sqsubseteq f$.
- f is strongly Weihrauch reducible to g, $f \leq_{\mathrm{sW}} g$, if there are computable $H :\subseteq W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $HgK \sqsubseteq f$.
- Equivalences $f \equiv_{\mathrm{W}} g$ and $f \equiv_{\mathrm{sW}} g$ are defined as usual.

Theorem (Tavana and Weihrauch 2011)

$f \leq_{\mathrm{W}} g \iff$ *there is a Turing machine that computes f and uses g as an oracle exactly once during its infinite computation.*

Examples of Mathematical Problems

- The Limit Problem is the mathematical problem

$$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, ...\rangle \mapsto \lim_{i\to\infty} p_i$$

  with $\mathrm{dom}(\lim) := \{\langle p_0, p_1, ...\rangle : (p_i)_i \text{ is convergent}\}$.

- Martin-Löf Randomness is the mathematical problem $\mathsf{MLR} : 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ with

$$\mathsf{MLR}(x) := \{y \in 2^{\mathbb{N}} : y \text{ is Martin-Löf random relative to } x\}.$$

- Weak Weak Kőnig's Lemma is the mathematical problem

$$\mathsf{WWKL} :\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$$

  with $\mathrm{dom}(\mathsf{WWKL}) := \{T \in \mathrm{Tr} : \mu([T]) > 0\}$.

- The Intermediate Value Theorem is the problem

$$\mathsf{IVT} :\subseteq \mathrm{Con}[0,1] \rightrightarrows [0,1], f \mapsto f^{-1}\{0\}$$

  with $\mathrm{dom}(\mathsf{IVT}) := \{f : f(0) \cdot f(1) < 0\}$.

- The Zero Problem $\mathsf{Z}_X :\subseteq \mathcal{C}(X) \rightrightarrows X, f \mapsto f^{-1}\{0\}$.
- The Choice Problem $\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$.

Algebraic Operations

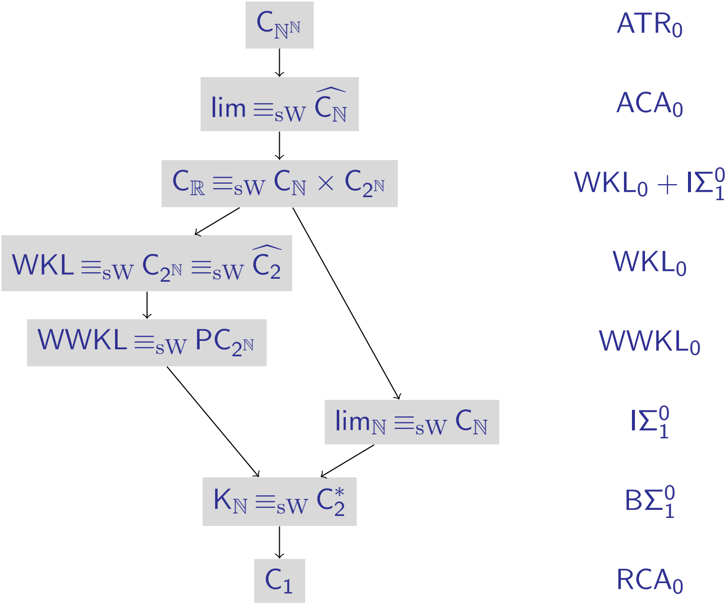
Definition

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ we define:

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z$, $(x, w) \mapsto f(x) \times g(w)$ (Product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z$, $z \mapsto \begin{cases} f(z) \text{ if } z \in X \\ g(z) \text{ if } z \in W \end{cases}$ (Coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z$, $(x, w) \mapsto f(x) \sqcup g(w)$ (Sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*$, $f^* = \bigsqcup_{i=0}^{\infty} f^i$ (Star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$, $\widehat{f} = \mathsf{X}_{i=0}^{\infty} f$ (Parallelization)

- Weihrauch reducibility induces a lattice with the coproduct $\sqcup$ as supremum and the sum $\sqcap$ as infimum.
- Parallelization and star operation are closure operators in the Weihrauch lattice.

Basic Complexity Classes and Reverse Mathematics



Compositional Product and Implication

The Weihrauch lattice is not complete and infinite suprema and infima do not always exist. There are some known existent ones.

Definition

For two mathematical problem f, g we define

- $f * g := \max\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f,\ g_0 \leq_{\mathrm{W}} g\}$ compos. product
- $g \rightarrow f := \min\{h : f \leq_{\mathrm{W}} g * h\}$ implication

Theorem (B. and Pauly 2016)

The compositional product $f * g$ and the implication $g \rightarrow f$ exist for all problems f, g.

Martin-Löf Randomness

Proposition (B., Gherardi and Hölzl 2015)

$\mathsf{MLR} * \mathsf{MLR} \leq_{\mathrm{W}} \mathsf{MLR}$

Proof. This is a consequence of van Lambalgen's Theorem. □

Corollary

The class of functions $f \leq_{\mathrm{W}} \mathsf{MLR}$ is closed under composition.

Theorem (B. and Pauly 2016)

$\mathsf{MLR} \equiv_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$.

Proof. $(\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL}) \leq_{\mathrm{W}} \mathsf{MLR}$: It suffices to prove $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * \mathsf{MLR}$, which follows from Kučera's Lemma.

$\mathsf{MLR} \leq_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$: Given some h with $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * h$ we need to prove that $\mathsf{MLR} \leq_{\mathrm{W}} h$. Given some universal Martin-Löf test $(U_i)_i$, we use $A_0 := 2^{\mathbb{N}} \setminus U_0$ and the fact that Martin-Löf randoms are stable under finite changes. □

Further Notions of Randomness

Theorem (Hölzl and Miyabe 2015)

$\mathrm{WR} <_{\mathrm{W}} \mathrm{SR} \leq_{\mathrm{W}} \mathrm{CR} <_{\mathrm{W}} \mathrm{MLR} <_{\mathrm{W}} \mathrm{W2R} <_{\mathrm{W}} 2\text{-}\mathrm{RAN}$.

Proof. The strictness has been proved using hyperimmune degrees, high degrees and minimal degrees. □

- WR: Kurtz random
- SR: Schnorr random
- CR: computable random
- W2R: weakly 2-random
- n-RAN: n-random

Proposition (Bienvenu and Kuyper 2016)

$n\text{-}\mathrm{RAN} * n\text{-}\mathrm{RAN} \leq_{\mathrm{W}} n\text{-}\mathrm{RAN}$.

Proof. The proof is based on van Lambalgen's Theorem and generalized lowness properties. □

Quantitative Versions of WWKL

Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

By ε-WWKL $:\subseteq \text{Tr} \rightrightarrows 2^{\mathbb{N}}$ we denote the restriction of WKL to $\text{dom}(\varepsilon\text{-WWKL}) := \{T : \mu([T]) > \varepsilon\}$ for $\varepsilon \in \mathbb{R}$.

Theorem (DDHMS 2016 and B., Gherardi and Hölzl 2015)

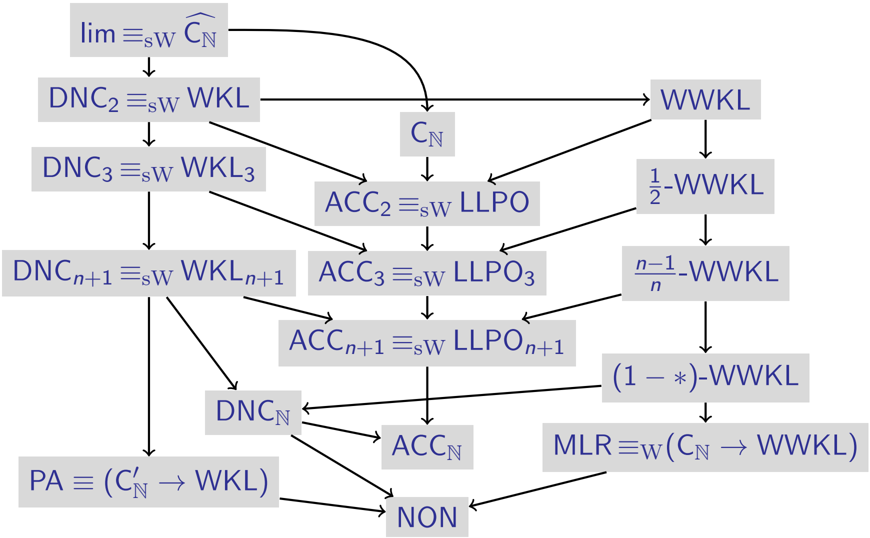
$\varepsilon\text{-WWKL} \leq_{\mathrm{W}} \delta\text{-WWKL} \iff \varepsilon \geq \delta$ *for all* $\varepsilon, \delta \in [0, 1]$.

Proof. (Idea) "$\Longrightarrow$" Assume $\varepsilon < \delta$. Then there are positive integers a, b with $\varepsilon < \frac{a}{b} \leq \delta$. We consider

- $C_{a,b}$ which is C_b restricted to sets $A \subseteq \{0, ..., b-1\}$ with $|A| \geq a$.

Then $C_{a,b} \leq_{\mathrm{W}} \varepsilon\text{-WWKL}$ and $C_{a,b} \not\leq_{\mathrm{W}} \delta\text{-WWKL}$. Hence $\varepsilon\text{-WWKL} \not\leq_{\mathrm{W}} \delta\text{-WWKL}$ □

Joint Results with Hendtlass and Kreuzer 2015



- $(1-*)\text{-WWKL} :\subseteq \text{Tr}^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, (T_i)_i \mapsto \bigsqcup_{i=0}^{\infty}(1-2^{-i})\text{-WWKL}(T_i)$

Jumps

- For every representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ we define the jump $\delta' :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ by $\delta' := \delta \circ \lim$.
- $X' = (X, \delta')$ denotes the corresponding represented space.
- For $f :\subseteq X \rightrightarrows Y$ we define its jump by $f' :\subseteq X' \rightrightarrows Y, x \mapsto f(x)$.
- For instance $\mathrm{id}' \equiv_{\mathrm{sW}} \lim$, $\mathsf{WKL}' \equiv_{\mathrm{sW}} \mathsf{KL} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$, etc.
- $n\text{-}\mathsf{RAN} \equiv_{\mathrm{sW}} \mathsf{MLR}^{(n-1)}$.

Proposition (B., Gherardi and Marcone 2012)

$f \leq_{\mathrm{sW}} g \Longrightarrow f' \leq_{\mathrm{sW}} g'$ and $f \leq_{\mathrm{sW}} f'$.

- $f <_{\mathrm{W}} f'$ does not hold in general: $f \equiv_{\mathrm{sW}} f'$ for a constant f.
- $f <_{\mathrm{W}} g$ is compatible with: $f' \equiv_{\mathrm{W}} g'$, $f' <_{\mathrm{W}} g'$, $g' <_{\mathrm{W}} f'$, $f' \,|_{\mathrm{W}}\, g'$.

Theorem (B., Hölzl and Kuyper 2016)

$f' \leq_{\mathrm{W}} g' \Longrightarrow f \leq_{\mathrm{W}} g$ with respect to the halting problem.

Uniform Theorem of Kurtz

Theorem of Kurtz. Every 2–random computes a 1–generic.

Theorem (B., Hendtlass and Kreuzer 2015)

$1\text{-GEN} <_{\mathrm{W}} (1-*)\text{-WWKL}'$.

Proof. (Idea) We apply the "fireworks technique" of Rumyantsev and Shen to get a uniform reduction. □

Theorem (B., Hendtlass and Kreuzer 2015)

$\mathrm{BCT}_0' \not\leq_{\mathrm{W}} \mathrm{WWKL}^{(n)}$ *for all* $n \in \mathbb{N}$.

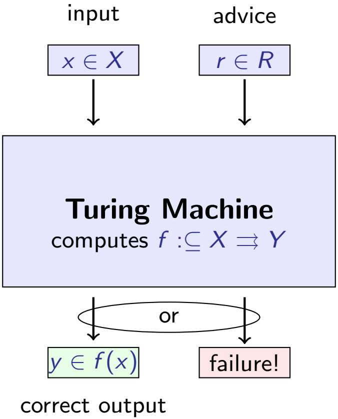
Proof. (Idea) There exists a co-c.e. comeager set $A \subseteq 2^{\mathbb{N}}$ such that no point of A is low for Ω. $\mathrm{WWKL}^{(n)}$ has a realizer that maps computable inputs to outputs that are low for Ω for $n \geq 1$. □

Corollary

$\mathrm{BCT}_0' \not\leq_{\mathrm{W}} 1\text{-GEN}$.

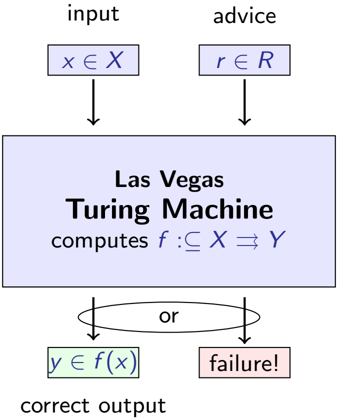
Las Vegas Computability

Turing Machines with Advice



Condition: $(\forall x \in \mathrm{dom}(f))\ \{r \in R : r \text{ does not fail with } x\} \neq \emptyset$

Las Vegas Turing Machines



Condition: $(\forall x \in \mathrm{dom}(f))\ \mu\{r \in R : r \text{ does not fail with } x\} > 0$

Calibrating Computability with Choice

Theorem (B., de Brecht and Pauly 2012)

For $R \subseteq \mathbb{N}^{\mathbb{N}}$ *and* $f :\subseteq X \rightrightarrows Y$ *the following are equivalent:*

- $f \leq_{\mathrm{W}} \mathsf{C}_R$,
- f *is computable on a Turing machine with advice from* R.

Corollary

- $f \leq \mathsf{C}_1 \iff f$ *is computable,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \iff f$ *comp. with finitely many mind changes,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} \iff f$ *is non-deterministically computable,*
- $f \leq_{\mathrm{W}} \mathsf{PC}_{2^{\mathbb{N}}} \iff f$ *is Las Vegas computable,*
- $f \leq_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} \iff f$ *is limit computable,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \iff f$ *is effectively Borel measurable.*

In the last case f *is single-valued on computable Polish spaces.*

Independent Choice Theorem

Theorem (B., de Brecht and Pauly 2012)

$\mathsf{C}_R * \mathsf{C}_S \leq_{\mathrm{W}} \mathsf{C}_{R \times S}$ *for all* $R, S \subseteq \mathbb{N}^{\mathbb{N}}$.

Proof. Run a Turing machine that simulates upon advice (r, s) two consecutive machines with advice r and s, respectively. □

Proposition

If $s : R \to S$ *is a computable surjection, then* $\mathsf{C}_S \leq_{\mathrm{W}} \mathsf{C}_R$.

Corollary

C_R *is closed under composition for* $R \in \{\mathbb{N}, 2^{\mathbb{N}}, \mathbb{N} \times 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}\}$.

Corollary (Gherardi and Marcone 2009, B. and Gherardi 2011)

WKL *is closed under composition.*

Independent Choice Theorem

Theorem (B., Gherardi and Hölzl 2015)

$\mathsf{PC}_R * \mathsf{PC}_S \leq_{\mathrm{W}} \mathsf{PC}_{R\times S}$ *for* $R, S \subseteq \mathbb{N}^{\mathbb{N}}$ *with* σ*–finite Borel measures and their product measure.*

Proof. (Sketch) The proof proceeds along the lines of the case for closed choice plus an additional invocation of Fubini's Theorem. □

Corollary

PC_R *is closed under composition for* $R \in \{\mathbb{N}, 2^{\mathbb{N}}, \mathbb{N} \times 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}\}$.

Corollary

WWKL *is closed under composition.*

Corollary

Las Vegas computable functions are closed under composition.

Compositions of higher versions of WWKL

Theorem (Bienvenu and Kuyper 2016)

$\mathsf{WWKL}' * \mathsf{WWKL}' \equiv_{\mathrm{W}} \mathsf{PC}'_{2^{\mathbb{N}}} * \mathsf{PC}'_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PC}'_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{PC}'_{\mathbb{R}} * \mathsf{PC}'_{\mathbb{R}}$.

- This contrasts $\mathsf{WKL}' * \mathsf{WKL}' \equiv_{\mathrm{W}} \mathsf{WKL}''$.

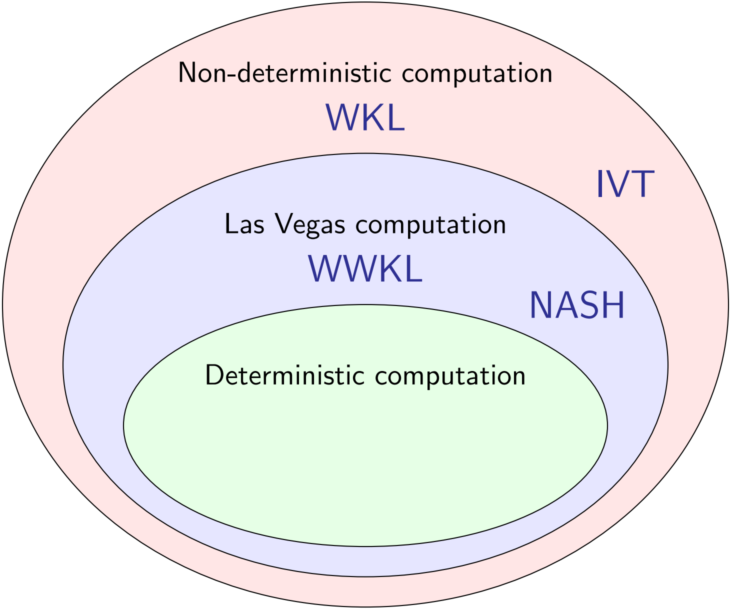
Proposition

- $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \not\leq_{\mathrm{sW}} \mathsf{WWKL}$,
- $\mathrm{id}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{WWKL}$,
- $\mathrm{id}_{F} \leq_{\mathrm{sW}} \mathsf{WWKL}$,

where $F := \{p \in 2^{\mathbb{N}} :$ p *contains only finitely many* 1*'s*$\}$.

Probabilistic Algorithms

Classes of Computability



Nash Equilibria

- A bi-matrix game is a pair $A, B \in \mathbb{R}^{m \times n}$ of $m \times n$–matrices.
- A vector $s = (s_1, ..., s_m) \in \mathbb{R}^m$ with $s_i \geq 0$ for all $i = 1, ..., m$ and $\sum_{j=1}^{m} s_j = 1$ is called a mixed strategy.
- By S^m we denote the set of mixed strategies of dimension m.
- A Nash equilibrium is a pair $(x, y) \in S^n \times S^m$ such that
  1. $x^{\mathrm{T}} Ay \geq w^{\mathrm{T}} Ay$ for all $w \in S^n$ and
  2. $x^{\mathrm{T}} By \geq x^{\mathrm{T}} Bz$ for all $z \in S^m$.
- Nash (1951) proved that for any bi-matrix game there exists a Nash equilibrium.
- By $\mathsf{NASH}_{n,m} : \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ we denote the mathematical problem, where $\mathsf{NASH}_{n,m}(A, B)$ is the set of all (x, y) such that (x, y) is a Nash equilibrium for (A, B).
- By $\mathsf{NASH} := \bigsqcup_{n,m \in \mathbb{N}} \mathsf{NASH}_{n,m}$ we denote the coproduct of all such games for finite $m, n \in \mathbb{N}$.

Theorem (Arno Pauly 2010)

$\mathsf{NASH} \equiv_{\mathrm{W}} \mathsf{RDIV}^*$.

A Las Vegas Algorithm for Robust Division

Proposition

Robust division RDIV *is Las Vegas computable.*

Proof.

1. Given $x, y \in [0, 1]$ and a random advice $r \in [0, 1]$, we aim to compute the fraction $z = \frac{x}{\max(x,y)}$.
2. We guess that r is a correct solution, i.e., $r = z$ if $y > 0$, and we produce approximations of r (rational intervals $(a, b) \ni r$).
3. Simultaneously, we try to find out whether $y > 0$, which we will eventually recognize, if this is correct.
4. If we find that $y > 0$, then we can compute the true result $z = \frac{x}{\max(x,y)}$ and produce approximations of it.
5. If at some stage we find that the best approximation (a, b) of r that was already produced as output is incompatible with z, i.e., if $z \notin (a, b)$, then we indicate a failure.

□

Nash Equilibria

Corollary

$\mathsf{NASH} \leq_{\mathrm{W}} \mathsf{WWKL}$.

Theorem

$\mathsf{NASH} \not\leq_{\mathrm{W}} \mathsf{IVT}$

Proof.

(Sketch) It is easy to see that $\mathsf{RDIV} \leq_{\mathrm{W}} \mathsf{IVT}$. However, one can prove (using a topological argument mixed with some combinatorial reasoning) that

$$\mathsf{C}_2 \times \mathsf{RDIV} \not\leq_{\mathrm{W}} \mathsf{IVT}.$$

Since $\mathsf{C}_2 \leq_{\mathrm{W}} \mathsf{RDIV}$, this implies $\mathsf{NASH} \equiv_{\mathrm{W}} \mathsf{RDIV}^* \not\leq_{\mathrm{W}} \mathsf{IVT}$. $\square$

A Probabilistic Algorithm for Zero Finding

1. A continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) \cdot f(1) < 0$ is given as input.
2. Guess a binary sequence or, equivalently, a bit $b \in \{0,1\}$ and a point $x \in [0,1]$.
3. Interpret the guess $b = 1$ such that the zero set $f^{-1}\{0\}$ contains no open intervals and use the trisection method to compute a zero $z \in [0,1]$ with $f(z) = 0$ in this case (disregarding x).
4. Interpret the guess $b = 0$ such that the zero set $f^{-1}\{0\}$ does contain an open interval and check whether $f(x) = 0$ in this case. Stop after finite time if this test fails and output x otherwise.

Warning: This is not a Las Vegas algorithm! But it yields:

Theorem

$\mathrm{IVT} \leq_{\mathrm{W}} \mathrm{WWKL}'$.

There is no Las Vegas Algorithm for Zero Finding

Theorem

$\mathsf{IVT} \not\leq_{\mathrm{W}} \mathsf{WWKL}$.

Proof.

(Idea) The proof is based on a finite extension construction: under the assumption that there is an algorithm for the reduction, one can create an instance (a function f) by finite extension that forces the reduction to translate this function into a tree that has measure zero. □

The inverse result $\mathsf{WWKL} \not\leq_{\mathrm{W}} \mathsf{IVT}$ is easy to see. Hence

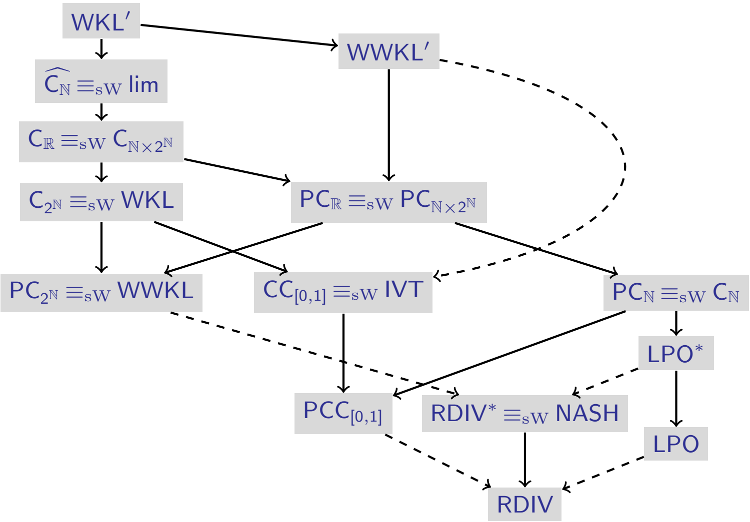
Corollary

$\mathsf{IVT} \mid_{\mathrm{W}} \mathsf{WWKL}$.

Corollary

$\mathsf{IVT} \mid_{\mathrm{W}} \mathsf{NASH}$.

From RDIV to WKL′ in the Weihrauch Lattice



Vitali Covering Theorem

Vitali Covering Theorem

- A point $x \in \mathbb{R}$ is captured by a sequence $\mathcal{I} = (I_n)_n$ of open intervals, if for every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ with $\mathrm{diam}(I_n) < \varepsilon$ and $x \in I_n$.
- $\mathcal{I}$ is a Vitali cover of $A \subseteq \mathbb{R}$, if every $x \in A$ is captured by $\mathcal{I}$.
- $\mathcal{I}$ eliminates A, if the I_n are pairwise disjoint and $\lambda(A \setminus \bigcup \mathcal{I}) = 0$ (where λ denotes the Lebesgue measure).

Theorem (Vitali Covering Theorem)

If $\mathcal{I}$ is a Vitali cover of $[0, 1]$, then there exists a subsequence $\mathcal{J}$ of $\mathcal{I}$ that eliminates $[0, 1]$.

Vitali Covering Theorem

Theorem (Brown, Giusto and Simpson 2002)

Over RCA_0 *the Vitali Covering Theorem is equivalent to Weak Weak Kőnig's Lemma* WWKL_0.

- Weak Weak Kőnig's Lemma is Weak Kőnig's Lemma restricted to trees whose set of infinite paths has positive measure.

Theorem (Diener and Hedin 2012)

Using intuitionistic logic (and countable and dependent choice) the Vitali Covering Theorem is equivalent to Weak Weak Kőnig's Lemma WWKL.

Vitali Covering Theorem

- $\mathcal{I}$ is called saturated, if $\mathcal{I}$ is a Vitali cover of $\bigcup \mathcal{I} = \bigcup_{n=0}^{\infty} I_n$.

Definition (Contrapositive versions of the Vitali Covering Theorem)

- VCT_0: Given a Vitali cover $\mathcal{I}$ of $[0,1]$, find a subsequence $\mathcal{J}$ of $\mathcal{I}$ that eliminates $[0,1]$.
- VCT_1: Given a saturated $\mathcal{I}$ that does not admit a subsequence that eliminates $[0,1]$, find a point that is not covered by $\mathcal{I}$.
- VCT_2: Given a sequence $\mathcal{I}$ that does not admit a subsequence that eliminates $[0,1]$, find a point that is not captured by $\mathcal{I}$.

- $\mathsf{VCT}_0 : (A \wedge B) \to C$,
- $\mathsf{VCT}_1 : (B \wedge \neg C) \to \neg A$,
- $\mathsf{VCT}_2 : \neg C \to \neg(A \wedge B)$.

Vitali Covering Theorem

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Definition (Contrapositive versions of the Vitali Covering Theorem)

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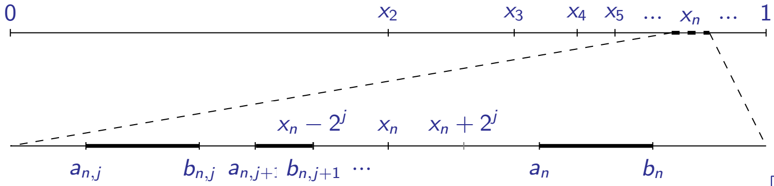
- VCT_2: Given a sequence $\mathcal{I}$ that does not admit a subsequence that eliminates $[0,1]$, find a point that is not captured by $\mathcal{I}$.

Theorem (B., Gherardi, Hölzl and Pauly 2016)

- VCT_0 *is computable,*
- $\mathrm{VCT}_1 \equiv_{\mathrm{sW}} \mathrm{WWKL}$,
- $\mathrm{VCT}_2 \equiv_{\mathrm{sW}} \mathrm{WWKL} \times \mathrm{C}_{\mathbb{N}}$.

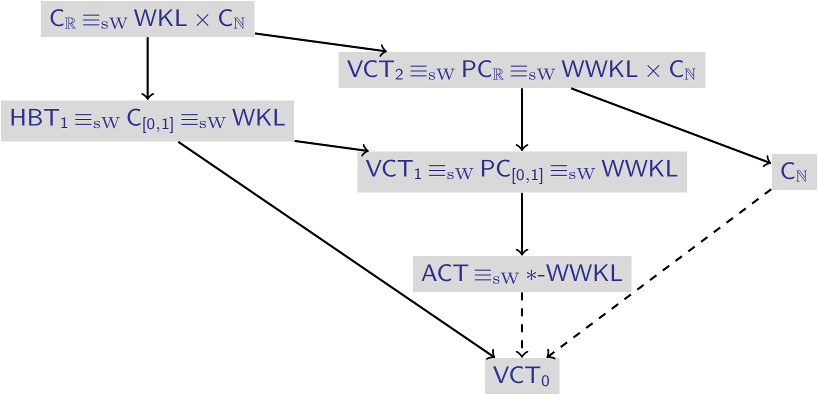
Proof.

- The proof of computability of VCT_0 is based on a construction that repeats steps of the classical proof of the Vitali Covering Theorem (and is not just based on a waiting strategy).
- The proof of $\mathsf{VCT}_1 \equiv_{\mathrm{sW}} \mathsf{WWKL}$ is based on the equivalence chain $\mathsf{VCT}_1 \equiv_{\mathrm{sW}} \mathsf{PC}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{WWKL}$.
- We use a Lemma by Brown, Giusto and Simpson on "almost Vitali covers" in order to prove $\mathsf{VCT}_2 \leq_{\mathrm{sW}} \mathsf{WWKL} \times \mathsf{C}_{\mathbb{N}}$. The harder direction is the opposite one for which it suffices to show $\mathsf{C}_{\mathbb{N}} \times \mathsf{VCT}_2 \leq_{\mathrm{sW}} \mathsf{VCT}_2$ by an explicit construction:



□

Vitali Covering Theorem in the Weihrauch Lattice



- $*\text{-WWKL} :\subseteq \mathbb{N} \times \text{Tr} \rightrightarrows 2^\mathbb{N}, (n, T) \mapsto \text{WWKL}(T)$ with $\text{dom}(*\text{-WWKL}) = \{(n, T) : \mu([T]) > 2^{-n}\}$.

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