Projection Operators in Computable Analysis

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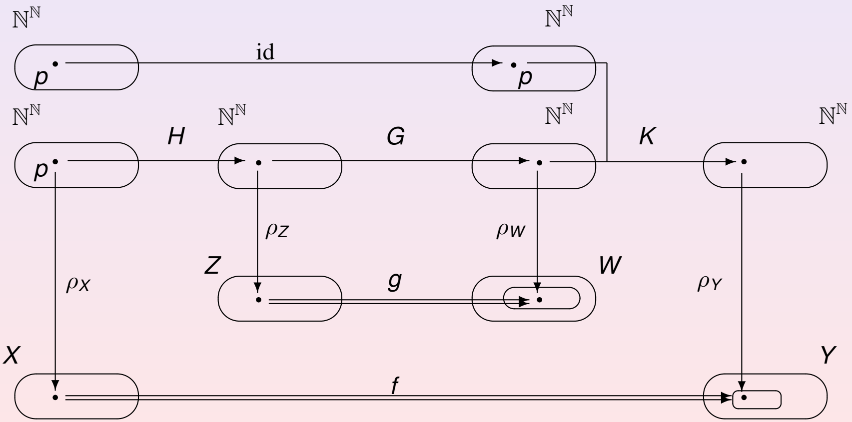
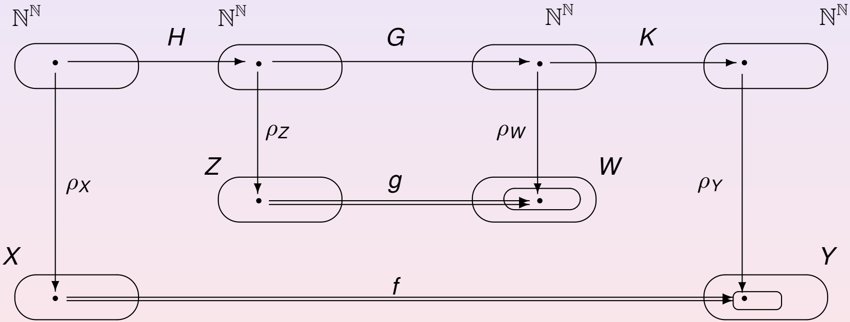
Preliminaries

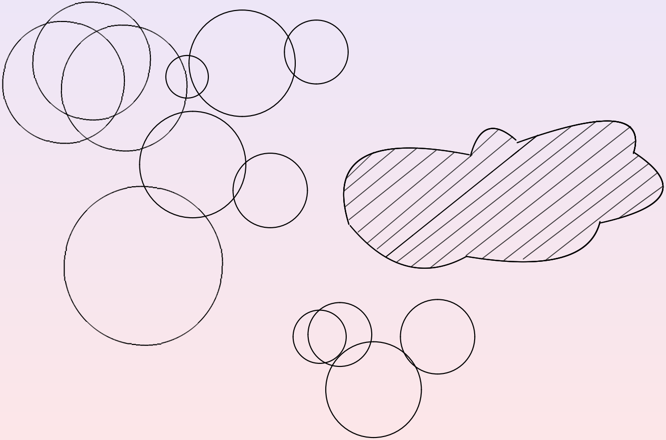
Weihrauch reducibility $f \leq_W g$



Strong Weihrauch reducibility $f \leq_{\mathrm{sW}} g$



Negative representation ψ_- for closed sets in computable metric spaces X

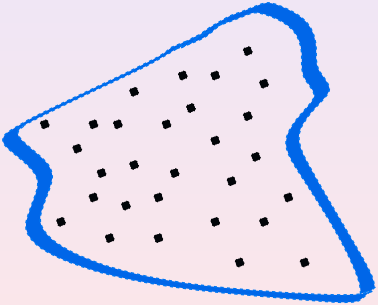


A list of rational open balls whose union gives the complement (here $X = \mathbb{R}^2$)

Negative representation κ_- for closed sets in computable metric spaces X

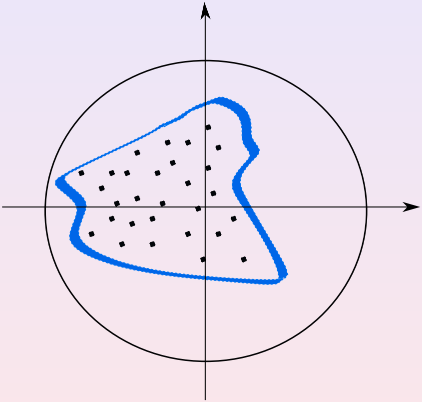
A list of all finite rational open coverings of the set

Positive representation ψ^+ for closed sets in complete computable metric spaces X



A dense sequence in the set (here $X = \mathbb{R}^2$)

Positive representation κ^+ for compact sets in $\mathbb{R}^n$



A dense sequence in the set and as a bound
a closed rational sphere centered in the origin
(here $n = 2$)

For a computable metric space

$$\mathrm{C}_X^- :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A)$$

is the (negative) closed choice operator.

For a computable metric space

$$\mathrm{K}_X^- :\subseteq \mathcal{K}_-(X) \rightrightarrows X, K \mapsto K)$$

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Obviously the corresponding operators for positive and total information (in complete computable metric spaces) are trivial.

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Important Weihrauch degress classes

id	computable
LLPO	constructive De Morgan
$C_{\mathbb{N}}^{-}$	computable with finitely many mind changes
WKL	non deterministic/ weakly computable
$C_{\mathbb{R}}^{-}$	non deterministic/weakly computable with finitely many mind changes
lim	limit computable
$\mathrm{BWT}_{\mathbb{R}}$	non deterministic/weakly limit computable

Exact projection operators

Definition

For $\mathbb{R}^n$ we define

- $\mathrm{ProjC}^{-}_{\mathbb{R}^n} :\subseteq \mathbb{R}^n \times \mathcal{A}_{-}(\mathbb{R}^n) \rightrightarrows \mathbb{R}^n$ be the (exact) negative closed projection operator such that

$$\mathrm{ProjC}^{-}_{\mathbb{R}^n}(x, A) = \{y \in A : d(x, y) = d(x, A)\},$$

  for $x \in \mathbb{R}^n$ and $A \in \mathcal{A}_{-}(\mathbb{R}^n)$;

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In an analoguous way we define the (exact) positive and the (exact) total projection operators for

$$\mathcal{A}^+(\mathbb{R}^n),\ \ \mathcal{K}^+(\mathbb{R}^n),\ \ \mathcal{A}(\mathbb{R}^n),\ \ \mathcal{K}(\mathbb{R}^n)$$

Theorem

$\mathrm{ProjK}^{-}_{\mathbb{R}^n} \equiv_{\mathrm{sW}} \mathrm{ProjC}^{-}_{\mathbb{R}^n}$, $\mathrm{ProjK}^{+}_{\mathbb{R}^n} \equiv_{\mathrm{sW}} \mathrm{ProjC}^{+}_{\mathbb{R}^n}$, $\mathrm{ProjK}_{\mathbb{R}^n} \equiv_{\mathrm{sW}} \mathrm{ProjC}_{\mathbb{R}^n}$.

Theorem

1. $\mathrm{ProjC}^{-}_{\mathbb{R}^n}$ *and* $\mathrm{ProjC}^{+}_{\mathbb{R}^n}$ *are weakly limit computable, that is* $\mathrm{ProjC}^{-}_{\mathbb{R}^n}, \mathrm{ProjC}^{+}_{\mathbb{R}^n} \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$*.*
2. $\mathrm{ProjC}_{\mathbb{R}^n}$ *is weakly computable, that is* $\mathrm{ProjC}_{\mathbb{R}^n} \leq_{\mathrm{sW}} \mathsf{WKL}$*.*

PROOF:

- Total information: Given $x \in \mathbb{R}^n$ and $A \in \mathcal{A}(\mathbb{R}^n)$ compute $d(x, A) \in \mathbb{R}$.
- compute $A \cap \partial\overline{B}(x, d(x, A)) \in \mathcal{A}_{-}(\mathbb{R}^n)$
- take an $N > ||x|| + d(x, A)$. This bound is used to translate $A \cap \partial\overline{B}(x, d(x, A))$ into a computable element of $\mathcal{K}_{-}(\mathbb{R}^n)$
- select a point from this compact set by WKL
- Partial informations: use $\lim \times \lim \equiv_{\mathrm{W}} \lim$ to have $A \in \mathcal{A}(x)$ and $d(x, A) \in \mathbb{R}$.
- Proceed as above with respect to total information
- In conclusion, we compose a limit computable and a weakly computable function $(\mathrm{ProjC}^{-}_{\mathbb{R}^n}, \mathrm{ProjC}^{+}_{\mathbb{R}^n} \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}})$

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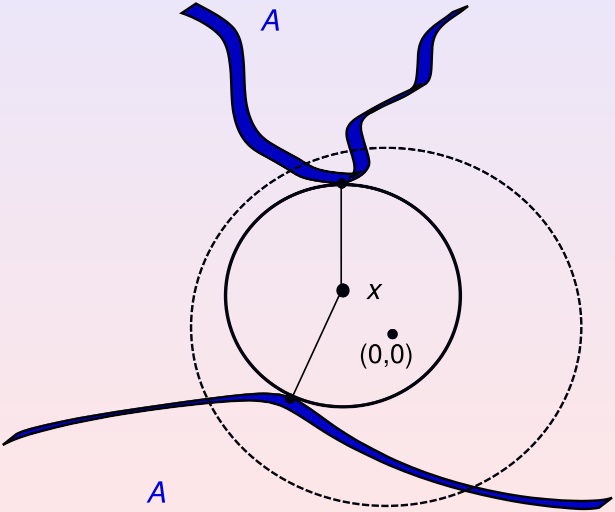
A
x
(0,0)
A

Open problem: Does $\mathrm{ProjC}^-_{\mathbb{R}^n} \equiv_{\mathrm{W}}$ $\mathrm{BWT}_{\mathbb{R}}$ hold?

Partial results:

Theorem

$\mathrm{BWT}_2, \lim <_{\mathrm{W}} \mathrm{ProjK}^-_{\mathbb{R}^n} \equiv_{\mathrm{sW}} \mathrm{ProjC}^-_{\mathbb{R}^n} \leq_{\mathrm{W}} \mathrm{BWT}_{\mathbb{R}}$.

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$\mathrm{ProjC}^-_{\mathbb{R}} <_{\mathrm{W}} \mathrm{BWT}$.

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$\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathrm{ProjC}^{+}_{\mathbb{R}^n}$ *for* $n \geq 2$.

Theorem

$\mathrm{ProjC}_{\mathbb{R}} \equiv_{\mathrm{W}}$ LLPO.

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Approximated projection operators

Definition

Let $\mathrm{ProjC}^{-}_{\varepsilon\,\mathbb{R}^n} : \mathbb{R}^n \times \mathcal{A}_{-}(\mathbb{R}^n) \rightrightarrows \mathbb{R}^n$ be the ε-approximated negative closed projection operator on X such that

$$\mathrm{ProjC}^{-}_{\varepsilon\,\mathbb{R}^n}(x, A) = \{y \in A \ : \ d(x, y) \leq (1 + \varepsilon)d(x, A)\},$$

for $x \in \mathbb{R}^n$, $A \in \mathcal{A}_{-}(\mathbb{R}^n)$ and for rational $0 < \varepsilon < 1$.

The (approximated) positive and total projections operators for closed sets are defined analoguously for $\mathcal{A}^{+}(\mathbb{R}^n)$ and $\mathcal{A}(\mathbb{R}^n)$ respectively.

The (approximated) negative, positive and total projections operators for compact sets can also be defined by replacing $\mathcal{A}_{-}(\mathbb{R}^n), \mathcal{A}_{+}(\mathbb{R}^n), \mathcal{A}(\mathbb{R}^n)$ through $\mathcal{K}_{-}(\mathbb{R}^n), \mathcal{K}_{+}(\mathbb{R}^n), \mathcal{K}(\mathbb{R}^n)$ correspondingly.

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Fact

$\mathrm{Proj}_\varepsilon \mathrm{K}^-_{\mathbb{R}^n} \leq_{\mathrm{sW}} \mathrm{Proj}_\varepsilon \mathrm{C}^-_{\mathbb{R}^n}$, $\mathrm{Proj}_\varepsilon \mathrm{K}^+_{\mathbb{R}^n} \leq_{\mathrm{sW}} \mathrm{Proj}_\varepsilon \mathrm{C}^+_{\mathbb{R}^n}$,
$\mathrm{Proj}_\varepsilon \mathrm{K}_{\mathbb{R}^n} \leq_{\mathrm{sW}} \mathrm{Proj}_\varepsilon \mathrm{C}_{\mathbb{R}^n}$.

Theorem

$\mathrm{Proj}_{\varepsilon} \mathrm{C}^{-}_{\mathbb{R}^n} \equiv_{\mathrm{sW}} \mathrm{C}^{-}_{\mathbb{R}}$

We use the space $\omega + 1$ with domain $\{-2^{-n}\}_{n\in\mathbb{N}} \cup \{0\}$ with topology and metric induced by $\mathbb{R}$, and representation

$$\mathrm{dom}(\rho_{\omega+1}) = \{p \in \mathbb{N}^{\mathbb{N}} \ : \ (\exists^{\leq 1} i) p(i) \neq 0\}$$

$$\rho_{\omega+1}(p) = \begin{cases} -2^{-n} & \text{if } p(n) \neq 0 \\ 0 & \text{if } (\forall i) p(i) = 0 \end{cases}.$$

We consider the function:

$$min^{-}_{\omega+1} :\subseteq \mathcal{A}^{-}(\omega + 1) \to \omega + 1, A \mapsto \min(A)$$

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$\mathrm{Proj}_\varepsilon \mathrm{C}^+_{\mathbb{R}^n} \equiv_{\mathrm{W}} \mathsf{min}^-_{\omega+1}$.

Theorem

$\mathrm{Proj}_\varepsilon \mathrm{C}_{\mathbb{R}^n}$ *is computable.*

Precision	Information	Euclidean dimension	Weihrauch degree
Exact	negative	$n \geq 1$	Bolzano-Weierstraß Theorem (?)
	positive	$n \geq 2$	Bolzano-Weierstraß Theorem
	total	$n = 1$	lesser limited principle of omniscience
		$n \geq 2$	Weak König's Lemma
Approx.	negative	$n \geq 1$	choice for real closed sets
	positive	$n \geq 1$	minimum for closed sets in $\omega + 1$
	total	$n \geq 1$	computable

Applications: Whitney Extension Theorems

Theorem (First Whitney's Extension Theorem)

Given any (nonempty and non total) closed set $A \subseteq \mathbb{R}^n$ there is a linear operator $\mathcal{E}_0^A : C(A) \to C(\mathbb{R}^n)$ mapping each continuous function $f \in C(A)$ to a total continuous extension $\mathcal{E}_0^A(f)$ such that $\mathcal{E}_0^A(f)_{|\mathbb{R}^n \setminus A} \in C^\infty(\mathbb{R}^n \setminus A)$.

Improvements with respect to Urysohn-Tietze Lemma:

- the operators $\mathcal{E}_0^A$ are linear (over $C(A)$)
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Theorem (Second Whitney's Extension Theorem)

Given any (nonempty and nontotal) closed set $A \subseteq \mathbb{R}^n$, there is a linear operator $\mathcal{E}_k^A$ that maps each sequence $(f^\alpha)_{|\alpha| \leq k}$ of continuous functions $f^\alpha \in C(A)$ (for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$) satisfying

$$f^\alpha(x) = \sum_{|\alpha+\beta| \leq k} \frac{f^{\alpha+\beta}(y)}{\alpha!}(x-y)^\beta + R_\alpha(x,y) \leq M$$

with

$$|R_\alpha(x,y)| \leq M\, d(x,y)^{k+1-|\alpha|}$$

to total extensions $\widehat{f^\alpha} : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ such that $\widehat{f^\alpha} = (\widehat{f^0})^\alpha = \hat{f}^\alpha$ (here $|\alpha| = \alpha_1 + ... + \alpha_n$).

Intuitively, $\mathcal{E}_k^A$ extends a sequence made of a continuous function f over A and its *potential* partial derivatives until the k-th degree to a total continuous extension $\hat{f} \in C^k(\mathbb{R}^n)$ of f and to its partial derivatives $\hat{f}^\alpha$ coinciding with f^α over A.

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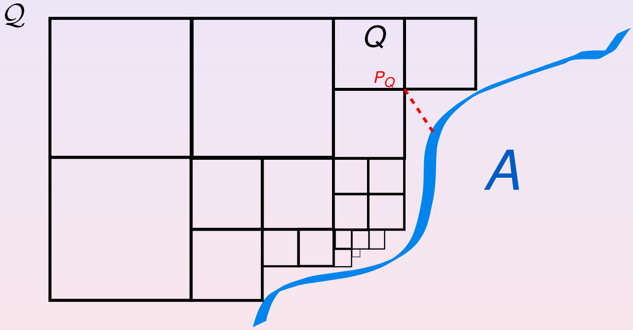
Classical proofs define $\mathcal{E}_0$ for instance following way [Ste70]:

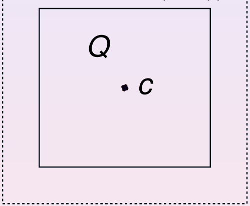
$$\mathcal{E}_0(f)(x) := \begin{cases} f(x) & \text{if } x \in A \\ \sum_{Q \in \mathcal{Q}} f(P_Q)\varphi_Q^*(x) & \text{else.} \end{cases}$$

Here:

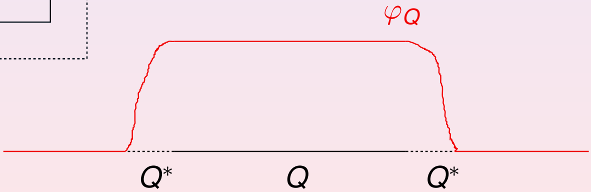
- $\mathcal{Q}$ is a family of cubes that constitutes a tiling of the whole complement of $\mathbb{R}^n \setminus A$,
- $\sum_{Q \in \mathcal{Q}} \varphi_Q^* = 1$ (partition of the unity)
- the points P_Q are *projection points* of cubes Q over A.


$\mathcal{Q}$
Q
P_Q
A

$$Q^* := (1+\varepsilon)(Q-c)+c$$



$$\varphi_Q^*(x) := \frac{\varphi_Q(x)}{\sum_{Q' \in \mathcal{Q}} \varphi_{Q'}(x)}$$



Analogously, classical proofs define $\mathcal{E}_k$ for instance in the following way [Ste70]:

$$\pi_\alpha(\mathcal{E}_k((f^\alpha(x))_{|\alpha|\leq k})(x) :=$$

$$\begin{cases} f^\alpha(x) & \text{for } x \in A, \\ \sum_{Q\in\mathcal{Q}} \frac{\partial^\alpha}{\partial x^\alpha} \sum_{|\alpha|\leq k} \frac{f^\alpha(x, P_Q)}{\alpha!}(x - P_Q)^\alpha \varphi_Q^*(x) & \text{else.} \end{cases}$$

Approximations are useful!

Let $C_c(\mathbb{R}^n) \subseteq C_p(\mathbb{R}^n) \times \mathcal{A}(x)$ be the space of all partial continuous functions with closed domain.
More precisely, $(f, A) \in C_c(\mathbb{R}^n)$ iff f is a partial function over $\mathbb{R}^n$ and $A = \mathrm{dom}(f)$.
$C_c(\mathbb{R}^n)$ is represented in the following way:

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Theorem (First computable Whitney Extension Theorem)

There exists a computable operator $\mathrm{WET}_0 : C_c(\mathbb{R}^n) \rightrightarrows C(\mathbb{R}^n)$ *that*

- *maps every input* $(f, A) \in C_c(\mathbb{R}^n)$ *to continuous total extensions* $\hat{f} \in C(\mathbb{R}^n)$ *of* f
- *is linear over* $C(A)$ *with respect to any given* ψ*-name of* A.

Approximations are useful!

Let $C_c(\mathbb{R}^n) \subseteq C_p(\mathbb{R}^n) \times \mathcal{A}(x)$ be the space of all partial continuous functions with closed domain.
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PROOF (scheme):
Preparation:

Given $(f, A) \in C_c(\mathbb{R}^n)$:

- *Effectivize* the construction of the tiling of $\mathbb{R}^n \setminus A$ by constructing a decidable collection of closed rational cubes $\mathcal{Q}$, such that $(\mathbb{R}^n \setminus A) = \bigcup \mathcal{Q}$;
- select *uniformly* for each $Q \in \mathcal{Q}$ a computable (smooth) continuous function φ_Q such that $\varphi_Q(z) = 1$ for all $z \in Q$ and $\varphi_Q(z) = 0$ for all $z \notin Q^*$;
- for each cube $Q \in \mathcal{Q}$ select a point R_Q that constitutes a *very rough (!) approximated projection* of Q over A. More precisely, any point $z \in A$ satisfying $d(z, Q) < 5\,\mathrm{diam}(Q)$ will work.

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Define

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \in A \\ \sum_{Q \in \mathcal{Q}} f(R_Q)\varphi_Q^*(x) & \text{else.} \end{cases}$$

Computation of $\hat{f}(x)$ for every $x \in \mathbb{R}^n$:

- Until we do not know whether $x \in A$, at stage i search for a basic open ball $B(y', \delta)$ such that
  - $x, y \in B(y', \delta)$ for some $y \in A$
  - $R_Q \in B(y', c\delta)$ for a fixed suitable computable constant c (e.g.: $c := \frac{28}{1-\varepsilon} + 3$)
  - $f(\overline{B}(y', \delta)) \subseteq f(\overline{B}(y', c\delta)) \subseteq B(y'', 2^{-i})$

  Then take y'' as the i-th rational approximation of $\hat{f}(x)$.
- If suddenly it turns out that $x \notin A$, then
  - replace *effectively* $\mathcal{Q}$ by the *finite* cube collection $\mathcal{Q}_x := \{Q \in \mathcal{Q} : x \in Q^*\}$;
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Theorem (Second computable Whitney Extension Theorem)

There is a computable multi-valued operator $\mathrm{WET}_k : \mathbb{J}^k \rightrightarrows C(\mathbb{R}^n)^k$ that maps every jet $(M, (f^\alpha)_{|\alpha|\le k})$ to sequences $(\widehat{f^\alpha})_{|\alpha|\le k}) \in C(\mathbb{R}^n)^k$ such that for all α with $|\alpha| \le k$ it holds

- *$\widehat{f^\alpha}$ is a total continuous extension of f^α,*
- *$\widehat{f^\alpha}$ is the α-partial derivative of $\widehat{f^0} = \hat{f}$.*

PROOF (sketch): We want to compute the extensions of *each* f^α so defined:

$$\widehat{f^\alpha}(x) := \begin{cases} f^\alpha(x) & \text{for } x \in A, \\ \sum_{Q\in\mathcal{Q}} \frac{\partial^\alpha}{\partial x^\alpha} \sum_{|\alpha|\le k} \frac{f^\alpha(x,R_Q)}{\alpha!}(x-R_Q)^\alpha \varphi^*(x) & \text{else} \end{cases}$$

- At stage s look for some point $y \in A$ such that for every α with $|\alpha| \le k$:
  1. $C_\alpha \cdot d(x,y)^{k+1-|\alpha|} < 2^{-s-|\alpha|}$, where C_α is a fixed suitable computable constant depending on α
  2. $y \in \mathrm{Proj}_{\frac{1}{2}}\mathrm{C}(x,A)$, where $\mathrm{Proj}_{\frac{1}{2}}\mathrm{C}(x,A) = \{y \in A | d(x,y) \le (1+\frac{1}{2})d(x,A)\}$
- For the first suitable y that is found compute (an approximation of) $P_\alpha(x,y) := \sum_{|\alpha+\beta|\le k} \frac{f^{\alpha+\beta}(y)}{\beta!}(x-y)^\beta$ for $|\beta| \le k$
- In case we realize that $x \notin A$, then we compute $f^\alpha(x)$ by using the R_Q's such that $x \in Q^*$

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The algorithm works because:

If $x \in A$, then $\mathrm{Proj}_{\frac{1}{2}} \mathrm{C}_{\mathbb{R}^n}(x, F) = x$, and then $f^\alpha(x) = P_\alpha(x, y)$, since $R_\alpha(x, y) = R_\alpha(x, x) = 0$.

If $x \notin A$ then for suitable values of C_α it will hold

$$|\hat{f}^\alpha(x) - P_\alpha(x, y)| \leq C_\alpha d(x, y)^{k+1-|\alpha|}.$$

To evaluate $|\hat{f}^\alpha(x) - P_\alpha(x, y)|$ and C_α we need $y \in \mathrm{Proj}_{\frac{1}{2}} \mathrm{C}_{\mathbb{R}^n}(x, A)$. For such a y it holds indeed $d(x, y) \leq \frac{3}{2} d(x, A) \leq \frac{3}{2} d(x, z)$ for all $z \in A$.

This is very useful, because during the calculations for the the term

$$\sum_{Q \in \mathcal{Q}_x} \sum_{|\beta+\gamma| \leq k} M(7e+1)^{k+1-|\beta+\gamma|} d(x, y)^{k+1-|\beta|} A_{\alpha-\beta} \cdot 7^{|\alpha-\beta|} \cdot d(x, R_Q)^{-|\alpha-\beta|}$$

will appear, and then we can express it merely in terms of $d(x, y)$ obtaining:

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