Hybrid representations as a tool for Complexity Theory

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Survey

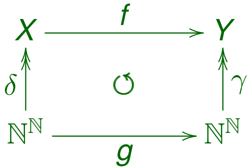
- Basics of Type-2 Complexity Theory
- Simple Complexity and CoPolish spaces
- Hybrid representations
- Admissibility for hybrid representations

Basics of Type 2 Complexity Theory

Remember

- A *representation* of X is a partial surjection $\delta\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow X$.
- Let $\delta\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow X$ and $\gamma\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow Y$ be representations.

  $f\colon X \to Y$ is called *(δ, γ)-computable*, if there is a computable function $g\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ s.t.



  commutes.
- $g\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ is *computable*, if there is an oracle Turing machine M that computes g.

Definition

Let M be an oracle machine that (δ, γ)-realizes $f\colon X \to Y$.

- Define the *computation time* of M by

$$Time_M(p, n) := \begin{cases} \text{the number of steps carried out by } M \\ \text{on input } (p, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \end{cases}$$

- Define the *relative computation time* on $A \subseteq X$ by

$$Time^{\delta}_M(A, n) := \sup \left\{ Time_M(p, n) \,\middle|\, \delta(p) \in A \right\}$$

Problem

- The sup may be equal to ∞, even if $A = \{x\}$.
- So M may not even have a time bound on singletons.

How can we ensure the existence of time bounds?

Lemma

- The computation time $\mathit{Time}_M \colon \mathrm{dom}(g_M) \times \mathbb{N} \to \mathbb{N}$ is continuous.
- For compact $L \subseteq \mathrm{dom}(g_M)$,

$$\mathit{Time}_M(L, n) := \sup \left\{ \mathit{Time}_M(p, n) \,\middle|\, p \in L \right\}$$

  exists (whenever M realizes a total function w.r.t. δ).

Summary

We can measure time on *compact* sets of *names*.

Observation

If δ is a continuous representation of a space X, then

$$\delta^{-1}[A] \text{ compact } \implies A \text{ compact.}$$

Definition

A continuous representation δ of X is called *proper*,
if $\delta^{-1}[K]$ is compact for every compact $K \subseteq \mathsf{X}$.

Lemma

For a proper δ, time complexity can be measured by a function

$$T\colon \{K \subseteq \mathsf{X} \mid K \text{ compact}\} \times \mathbb{N} \to \mathbb{N}$$

Example

The signed-digit representation ϱ_{sd} for $\mathbb{R}$ defined by

$$\varrho_{\mathrm{sd}}(p) := p(0) + \sum_{i=1}^{\infty} p(i) \cdot 2^{-i} \qquad \text{for } p \in \mathbb{Z} \times \{-1, 0, 1\}^{\mathbb{N}}$$

is proper.

Theorem

A sequential space X has a proper admissible representation iff X is separable metrisable.

Simple Complexity

Aim

Measurement of time complexity in terms of

- a *discrete* parameter on the input &
- the output precision.

Idea

- Equip δ with a "size function" $S\colon \mathrm{dom}(\delta) \to \mathbb{N}$.
- Measure time complexity by $T_M\colon \mathbb{N} \times \mathbb{N} \dashrightarrow \mathbb{N}$,

$$T_M(a, n) := \sup \{ Time_M(p, n) \mid S(p) = a \},$$

where M is a realizing machine.

Definition

We call $S\colon \mathrm{dom}(\delta) \to \mathbb{N}$ a *size function* for δ, if

- S is continuous,
- $S^{-1}\{a\}$ is compact for all $a \in \mathbb{N}$.

Example

Natural size functions for the signed-digit representation for $\mathbb{R}$:

- $S_1(p) = |p(0)|$
- $S_2(p) = \log_2(|p(0)| + 1)$

Proposition

δ has a size function iff $\mathrm{dom}(\delta)$ is locally-compact.

Lemma

Let δ be a representation with size function S. Then

$$T_M(a, n) = \sup \left\{ Time_M(p, n) \,\middle|\, S(p) = a \right\}$$

exists for all $a, n \in \mathbb{N}$ (whenever M realizes a total function on X).

Summary

Time complexity of a function f on (X, δ, S) can be measured in two *discrete* parameters:

- the size $S(p)$ of the input name p &
- the desired output precision.

Example

Let $\mathcal{P}$ be the vector space of polynomials.

- Suitable representation $\varrho_{\mathcal{P}}$:
  - Store the coefficients & an upper bound of the degree
- Size $S(q) \in \mathbb{N} \times \mathbb{N}$ of a name q:
  - the upper bound of the degree & the maximum of the integer parts of the coefficients
- Evaluation is $(\varrho_{\mathcal{P}}, \varrho_{\text{sd}}, \varrho_{\text{sd}})$-computable in polynomial time w.r.t. the size functions of $\varrho_{\mathcal{P}}$ and ϱ_{sd}.
- The final topology of $\varrho_{\mathcal{P}}$ does not have a countable base,
- but it is CoPolish.

CoPolish Spaces

Definition

We call a Hausdorff QCB-space X *CoPolish*, if $\mathbb{S}^X$ has a countable base.

Remark

- $\mathbb{S}$ denotes the Sierpiński space
- QCB-space $=$ a **q**uotient of a **c**ountably **b**ased top. space.
- QCB $=$ class of top. spaces which can be handled by TTE.

Proposition

Let X be a Hausdorff space with a countable base. Then:

- X is CoPolish $\iff$ X is locally compact.

Theorem (Characterisation)

Let X be a Hausdorff QCB-space. TFAE:

- X is CoPolish.
- $\mathbb{S}^{\mathsf{X}}$ is quasi-Polish.
- X is the direct limit of an increasing sequence of compact metrisable spaces.
- X has an admissible TTE-representation δ with a size function $S\colon \mathrm{dom}(\delta) \to \mathbb{N}$.
- X has an admissible TTE-representation with a locally-compact domain.

Proposition

- The category of CoPolish Hausdorff spaces
  - has finite products and equalisers (inherited from QCB),
  - but is not closed under forming QCB-exponentials.
- For any Y with a countable base and any CoPolish space X, Y^X has a countable base.
- [de Brecht & Sch.] For any (quasi-)Polish space Y and any CoPolish space X, Y^X is (quasi-)Polish.
- Hausdorff quotients of CoPolish Hausdorff spaces are CoPolish.

Locally convex vector spaces and CoPolishness

Theorem

Let $\mathfrak{X}$ be a locally convex vector space.

- $\mathfrak{X}$ is sep. metrisable $\Longleftrightarrow$ $\mathfrak{X}^*$ is CoPolish
- $\mathfrak{X}$ is CoPolish $\Longleftrightarrow$ $\mathfrak{X}^*$ is sep. metrisable $\Longleftrightarrow$ $\mathfrak{X}^*$ is Polish

Remark

$\mathfrak{X}^*$ denotes the dual of $\mathfrak{X}$ formed in QCB.

Example (CoPolish spaces)

- The Euclidean space $\mathbb{R}$.
- The space $\mathcal{P}$ of polynomials (equipped with the final topology of the aforementioned representation).
- The space $\mathcal{L}ip[0;1]$ of Lipschitz-continuous fcts on $[0;1]$.
- The space $\mathcal{A}$ of analytic functions on the unit interval.
- The space $\mathcal{E}^*$ of distributions over $\mathbb{R}$ with compact support.
- The space $\mathcal{S}^*$ of tempered distributions.
- The space $\mathcal{G}[0;1]$ of Gevrey functions on $[0;1]$.

Generalised size functions (parameter functions)

Definition

We call $\mu\colon \mathrm{dom}(\delta) \to \mathbb{N}^{\mathbb{N}}$ a *size function* for δ, if

- μ is continuous,
- the set $\{p \in \mathrm{dom}(\delta) \mid \mu(p) \leq_{\mathrm{pw}} \ell\}$ is compact for all $\ell \in \mathbb{N}^{\mathbb{N}}$.

Lemma

Let $\mu\colon \mathrm{dom}(\delta) \to \mathbb{N}^{\mathbb{N}}$ be continuous. TFAE:

- $\{p \in \mathrm{dom}(\delta) \mid \mu(p) \leq_{\mathrm{pw}} \ell\}$ is compact for every $\ell \in \mathbb{N}^{\mathbb{N}}$.
- $\mu^{-1}[K]$ is compact for any compact $K \subseteq \mathbb{N}^{\mathbb{N}}$ (i.e. μ is proper).

Lemma

Let δ be a representation with size function $\mu\colon \mathrm{dom}(\delta) \to \mathbb{N}^{\mathbb{N}}$.
Then

$$\mathcal{T}_M(\ell, n) := \sup \left\{ Time_M(p, n) \,\middle|\, \mu(p) \leq_{\mathrm{pw}} \ell \right\}$$

exists for all $\ell \in \mathbb{N}^{\mathbb{N}}$, $n \in \mathbb{N}$ (whenever M realizes a total function on X).

Corollary

Time complexity of a function f on (X, δ, μ) can be measured in:

- the size $\mu(p) \in \mathbb{N}^{\mathbb{N}}$ of the input name p &
- the desired output precision.

Hybrid Representations

Observation

Representations for spaces in Functional Analysis are typically constructed by encoding:

- a sequence of reals &
- a sequence of discrete information.

Definition

- Let $\mathbb{H} := [-1;1]^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$.
- A *hybrid representation* of X is a partial surjection $\psi\colon \mathbb{H} \dashrightarrow \mathsf{X}$.
- $f\colon \mathsf{X} \to \mathsf{Y}$ is $(\psi_{\mathsf{X}}, \psi_{\mathsf{Y}})$-computable, if there is a computable $h\colon \mathbb{H} \dashrightarrow \mathbb{H}$ s.t.

$$\begin{array}{ccc} \mathsf{X} & \xrightarrow{\quad f \quad} & \mathsf{Y} \\ \psi_{\mathsf{X}} \big\uparrow & \circlearrowleft & \big\uparrow \psi_{\mathsf{Y}} \\ \mathbb{H} & \xrightarrow{\quad h \quad} & \mathbb{H} \end{array}$$

Definition

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$$
\begin{array}{ccc}
\mathsf{X} & \xrightarrow{\quad f \quad} & \mathsf{Y} \\
\uparrow{\scriptstyle \psi_{\mathsf{X}}} & \circlearrowleft & \uparrow{\scriptstyle \psi_{\mathsf{Y}}} \\
\mathbb{H} & \xrightarrow{\quad h \quad} & \mathbb{H} \\
\uparrow{\scriptstyle \varrho_{\mathbb{H}}} & \circlearrowleft & \uparrow{\scriptstyle \varrho_{\mathbb{H}}} \\
\mathbb{N}^{\mathbb{N}} & \xrightarrow{\quad g \quad} & \mathbb{N}^{\mathbb{N}}
\end{array}
$$

Example (Hybrid representations)

- $\mathcal{C}[0;1]$:
  Choose a dense sequence $(d_i)_i$ in $[0;1]$. Let

$$\psi(r,p) = f \;:\Longleftrightarrow\; \begin{cases} \forall i \in \mathbb{N}.\, f(d_i) = r(i) \cdot p(0) \quad \& \\ k \mapsto p(k+1) \text{ is a modulus} \\ \qquad\qquad\qquad \text{of continuity for } f \end{cases}$$

- The space of polynomials $\mathcal{P}$:
  Let $\mathbb{H}_0 := [-1;1]^{\mathbb{N}} \times \mathbb{N}$ and

$$\psi(r, \langle c, d\rangle) = P \;:\Longleftrightarrow\; P(\mathrm{x}) = \sum_{k=0}^{d} c \cdot r(k) \cdot \mathrm{x}^k$$

- $\mathfrak{X}$ Banach space with Schauder basis e:

Remember

- *Schauder basis* of a Banach space $\mathfrak{X}$: a sequence $(e_i)_i$ of unit vectors s.t. every $x \in \mathfrak{X}$ can be written as

$$x = \sum_{i=0}^{\infty} x_i \cdot e_i \quad \text{for unique } (x_i)_i \in \mathbb{R}^{\mathbb{N}}.$$

- The coordinate functionals e_i^* are linear & continuous.
- There is a constant $C \in \mathbb{N}$ with $|e_i^*(x)| \leq C \cdot \|x\|$ for all x, i (provided that $\|e_j\| = 1$ for all j).

Example (contd.)

- $\mathfrak{X}$ Banach space with Schauder basis e and constant C. Let

$$\psi(r,p) = x \;:\Longleftrightarrow\; \begin{cases} \forall i \in \mathbb{N}.\ e_i^*(x) = r(i) \cdot p(0) \cdot C, \\ \|x\| \leq p(0), \\ \forall k \geq 1.\ \left\|x - \sum_{i=0}^{p(k)} e_i^*(x) \cdot e_i\right\| \leq 2^{-k} \end{cases}$$

Definition

Let M be an oracle machine realizing $f\colon (\mathsf{X}, \psi_{\mathsf{X}}) \to (\mathsf{Y}, \psi_{\mathsf{Y}})$.

A function $t\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 \to \mathbb{N}$ is a *time bound* for M, if

- for all $(r, p) \in \mathrm{dom}(\psi_{\mathsf{X}})$ and all $j, k \in \mathbb{N}$
- M produces $q(j)$ and some 2^{-k}-approximation to $s(j)$ (where (s, q) denotes the produced representative of the result)
- in $\leq t(p, j, k)$ steps.

Remark

- Hybrid representations ψ have as implicit size function $\mathrm{pr}_2\colon \mathbb{H} \to \mathbb{N}^{\mathbb{N}}$, $(r, p) \mapsto p$.
- It turns out to be reasonable to consider other size functions $\mu\colon \mathrm{dom}(\psi) \to \mathbb{N}^{\mathbb{N}}$.

Theorem

Any oracle Turing machine realising some function w.r.t. hybrid representations with closed domain has a continuous time bound $t\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 \to \mathbb{N}$.

The proof is based on:

Lemma

A hybrid representation ψ has a closed domain iff

$$\{(r,p) \in \mathrm{dom}(\psi) \mid p \in K\} \text{ is compact}$$

for every compact $K \subseteq \mathbb{N}^{\mathbb{N}}$.

Definition

A hybrid representation is *complete*, if its domain is closed.

Theorem

- A metrisable space has an admissible complete hybrid representation iff it is Polish.
- A Hausdorff QCB-space has an admissible complete hybrid representation over $\mathbb{H}_0 = [-1;1]^{\mathbb{N}} \times \mathbb{N}$ iff it is CoPolish.
- A quasi-normal space has an admissible complete hybrid representation iff it has a total admissible TTE-representation.

Remark

- Quasi-normal space $=$ QCB-space that arises as the sequentialisation of a normal space.
- Sep. metric spaces and CoPolish spaces are quasi-normal.
- Quasi-normal spaces have excellent closure properties:
  - cartesian closed
  - countable products and equalisers
  - countable co-products and co-equalisers

Theorem
The category of Hausdorff QCB-spaces having an admissible complete hybrid representation has

- countable products,
- countable co-products,
- equalisers.

But it is not closed under forming function spaces in QCB.

Admissibility for hybrid representations

Remember

Admissibility for TTE-representations guarantees:

$(*)$ Any continuous $f\colon X \to Y$ has a continuous realizer g.

Problem

Admissibility for TTE-representations does not work adequately for hybrid representations.

Aim

Find a notion of admissibility for hybrid representations s.t.

- $(*)$ holds,
- the $[-1;1]^{\mathbb{N}}$-part of $\mathbb{H}$ is used,
- natural hybrid representations are admissible.

Observation

In many examples, the $\mathbb{N}^{\mathbb{N}}$-part encodes a compact subset.

Example

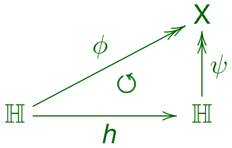
- $\mathcal{C}[0;1]$:
  $$K_p = \Big\{ f \,\Big|\, \|f\|_\infty \leq p(0) \;\&\; k \mapsto p(k+1) \text{ is a modulus of continuity for } f \Big\}$$
- $\mathcal{P}$, the space of polynomials:
  $$K_{\langle c,d \rangle} = \Big\{ x \mapsto \sum_{i=0}^{d} a_i \cdot x^i \,\Big|\, a_0, \ldots, a_d \in [-c;c] \Big\}$$
- Banach space $\mathfrak{X}$ with Schauder basis e:
  $$K_p = \Big\{ x \,\Big|\, \|x\| \leq p(0) \;\&\; \forall k \geq 1. \Big\| x - \sum_{i=0}^{p(k)} e_i^*(x) \cdot e_i \Big\| \leq 2^{-k} \Big\}$$

Remember

The compact subsets of a Hausdorff QCB-space X have a canonical representation κ_{X}.

Definition Let X be a Hausdorff QCB-space.

- Let HR_{X} be the set of all hybrid representations ϕ of X s.t.
  - ϕ is continuous,
  - there is a continuous $g\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathrm{dom}(\kappa_{\mathsf{X}})$ with $\phi(r,p) \in \kappa_{\mathsf{X}}(g(p))$ for all $(r,p) \in \mathrm{dom}(\phi)$.
- A hybrid representation ψ for X is $\mathbb{H}$-*admissible*, if
  - $\psi \in \mathrm{HR}_{\mathsf{X}}$,
  - any $\phi \in \mathrm{HR}_{\mathsf{X}}$ can be continuously translated into ψ, i.e. there is a continuous h s.t.



Proposition

Let $\psi_{\mathsf{X}}, \psi_{\mathsf{Y}}$ be $\mathbb{H}$-admissible hybrid reps for X, Y. TFAE:

- $f : \mathsf{X} \to \mathsf{Y}$ is continuous.
- f has a continuous $(\psi_{\mathsf{X}}, \psi_{\mathsf{Y}})$-realizer $h \colon \mathbb{H} \dashrightarrow \mathbb{H}$.
- f is $(\psi_{\mathsf{X}}, \psi_{\mathsf{Y}})$-computable relative to an oracle.

Example

- The examples from before.
- $\rho_{\mathrm{Float}} \colon [-1; 1] \times \mathbb{Z} \to \mathbb{R}$ with $\rho_{\mathrm{Float}}(r, z) := 2^z \cdot r$.

Non-Example

Most admissible TTE-reps δ (viewed as maps $(r, p) \mapsto \delta(p)$).

Proposition

Any quasi-normal space has an $\mathbb{H}$-admissible hybrid rep.

Summary

- Time bounds are guaranteed for compact sets of names.
- CoPolish Hausdorff spaces allow the measurement of complexity in discrete parameters.
- Important spaces are CoPolish, e.g. the duals of separable metrisable locally convex spaces.
- Hybrid representations yield a new approach to Complexity Theory for spaces in Functional Analysis.
- There exists a generalised notion of admissibility for hybrid representations.