A Limit Control Theorem with Applications

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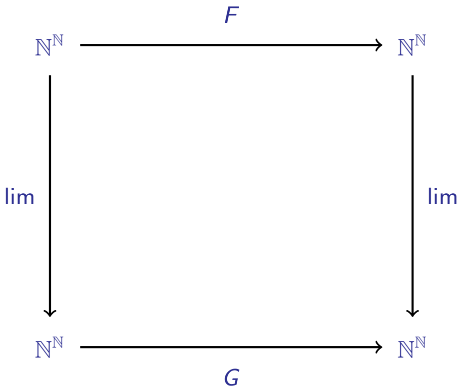
A Limit Diagram

We use the limit map on Baire space

- $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, \ldots\rangle \mapsto \lim_{i\to\infty} p_i$



How is continuity/computability of F and G related in this commutative diagram?

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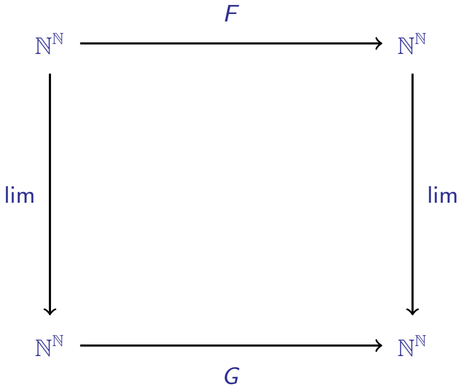

If $G : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is total and continuous/computable, then

$$F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, \ldots \rangle \mapsto \langle G(p_0), G(p_1), G(p_2), \ldots \rangle$$

is continuous/computable and satisfies the diagram.

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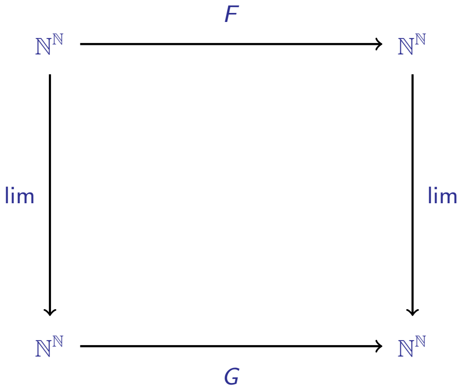

If $G^{\emptyset'} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is computable in the halting problem $\emptyset'$, then

$F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, \langle p_0, p_1, p_2, \ldots \rangle \mapsto \langle G^{\emptyset'[0]}(p_0), G^{\emptyset'[1]}(p_1), G^{\emptyset'[2]}(p_2), \ldots \rangle$

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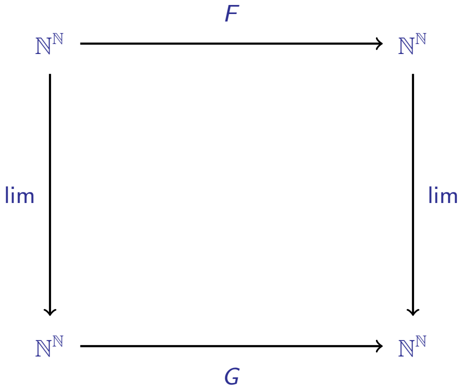

Let $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is computable (and potentially extensional).
Is there a suitable computable $G :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$?

Not in general!

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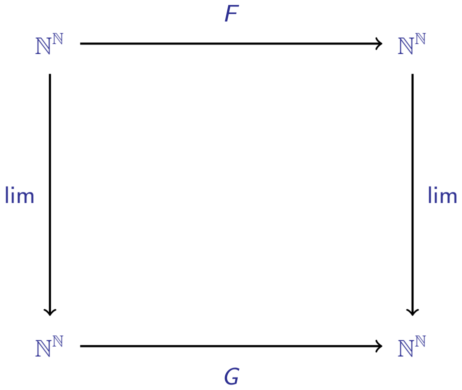

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A Galois Connection Between Limits and Jumps

We use the Turing jump operator on Baire space

- $\mathrm{J}:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}, p\mapsto p'$ where $p'(n):=\begin{cases}1 & \text{if } p\in U_n\\ 0 & \text{otherwise}\end{cases}$

Here $(U_n)_{n\in\mathbb{N}}$ is a standard enumeration of all c.e. open sets.

Theorem (B. 2007)

For $F:\subseteq\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ the following are equivalent:

1. *F is limit computable,*
2. *$F=\lim\circ G$ for some computable $G:\subseteq\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$,*
3. *$F=H\circ\mathrm{J}$ for some computable $H:\subseteq\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$.*

Caution (B., de Brecht, Pauly 2012):
One cannot replace computability by continuity!

Corollary

$\lim\equiv_{\mathrm{sW}}\mathrm{J}$.

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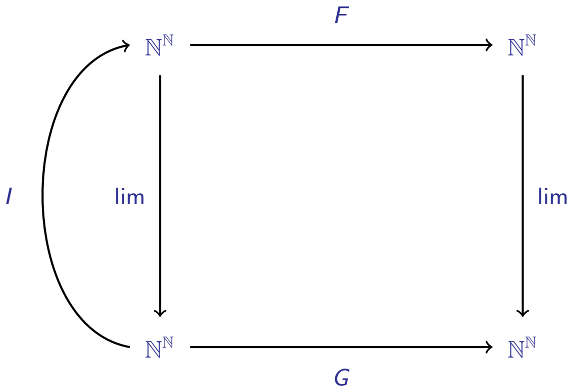

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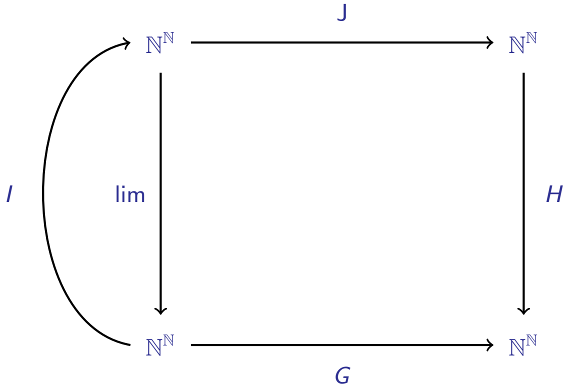

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Inverting Limits

Proposition (B., de Brecht and Pauly 2012)

The points of continuity of J are exactly the 1-generic points.

Proposition

There exists $I : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ that is computable in $\emptyset'$ such that

1. $\mathrm{J} \circ I$ *is computable in* $\emptyset'$,
2. $\lim \circ I = \mathrm{id}$.

Proof. The proof is remniscient of the proof of the Friedberg Jump Inversion Theorem. Given p, we have to find a sequence $I(p)$ so that $\lim \circ I(p) = p$ and $I(p)$ is 1-generic (i.e., a point of continuity of J). For this purpose $I(p)$ is constructed inductively such that longer and longer prefixes are chosen so that ∂U_n is avoided for all $n \in \mathbb{N}$. While $p \in U_n$ can be recognized, ensuring that $p \notin \overline{U_n}$ requires the halting problem $\emptyset'$. □

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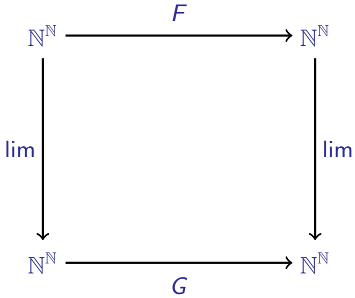
Theorem (B., Hendtlass and Kreuzer 2017)

For all computable $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *there exists* $G :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *computable in* $\emptyset'$ *so that* $G(p) \in \lim \circ F \circ \lim^{-1}(p)$.



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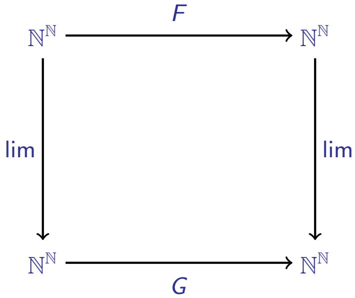

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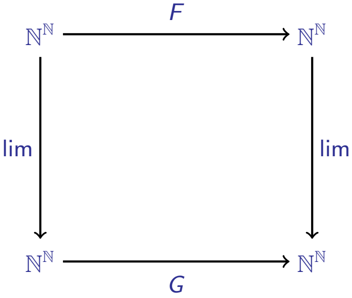

Corollary (written up somewhere?)

G *is computable in* $\emptyset'$ *iff it has a computable* $(\lim, \lim)$*–realizer* F.

A Galois Connection Between Limits and Jumps

We use the jump on derivative of a representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$

- $\delta' :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ with $\delta' := \delta \circ \lim$

and the integral

- $\int \delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ with $\int \delta := \delta \circ \mathsf{J}^{-1}$.

Corollary

For $F :\subseteq (X, \delta_X) \to (Y, \delta_Y)$ the following are equivalent:

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For representations $\delta_1, \delta_2 :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ we obtain

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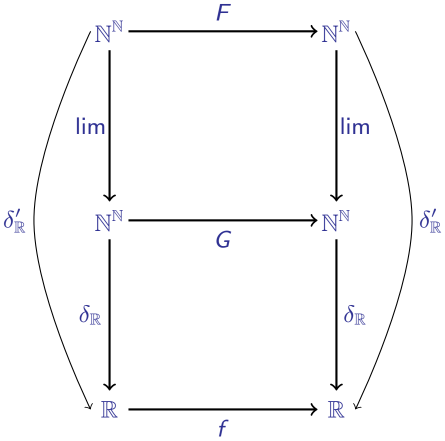
Theorem (B. and Hertling 2002)

A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is continuous with respect to the naive Cauchy representation if and only if it is continuous.



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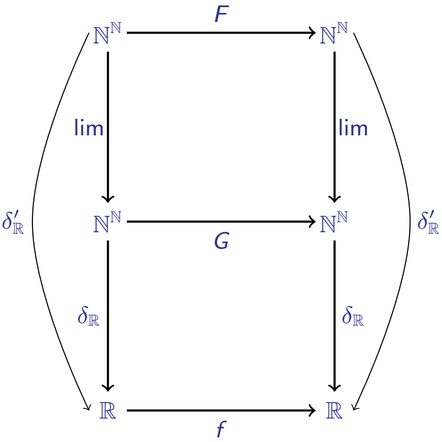

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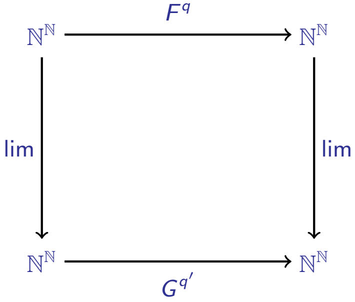
The Limit Control Theorem Relativized

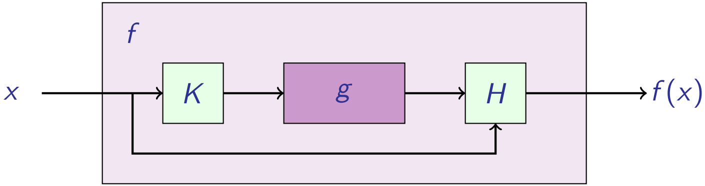
Theorem (B., Hendtlass and Kreuzer 2017)

For all $F^q :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *computable in* q *there is* $G^{q'} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *computable in* q' *so that* $G^{q'}(p) \in \lim \circ F^q \circ \lim^{-1}(p)$.



Weihrauch Reducibility

- For partial multi-valued functions f, g one defines Weihrauch reducibility $f \leq_{\mathrm{W}} g$:



  K, H are computable input and output adaptions, respectively.

- We define the compositional product

$$f * g := \max_{\leq_{\mathrm{W}}}\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f, g_0 \leq_{\mathrm{W}} g\}.$$

  This captures the most complicated problem that one can implement by first using g and then f (possibly after some intermediate computation).

Application 2: Decomposition of König's Lemma

For $f :\subseteq X \rightrightarrows Y$ we define the jump $f' :\subseteq X' \rightrightarrows Y$ (which is the same problem with a modified input representation).

Theorem (B., Hendtlass and Kreuzer 2017)

$\mathsf{WKL}' \equiv_{\mathrm{W}} \lim * \mathsf{COH}$.

- WKL is the problem: given an infinite binary tree T, find an infinite path $p \in [T]$.
- WKL′ is the problem: given a sequence $(T_i)_{i\in\mathbb{N}}$ of binary trees that converges to an infinite binary tree T, find an infinite path $p \in [T]$.
- COH is the problem: given a sequence $(R_i)_{i\in\mathbb{N}}$ of sets $R_i \subseteq \mathbb{N}$, find an infinite set $S \subseteq \mathbb{N}$ such that $S \subseteq^* R_i$ or $S \subseteq^* \mathbb{N} \setminus R_i$ for each $i \in \mathbb{N}$.

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Proof.

- $\mathsf{WKL}' \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$ (B., Gherardi, Marcone 2012).
- $\mathsf{BWT}_{\mathbb{R}}$ is the problem: given a sequence $(x_i)_{i\in\mathbb{N}}$ whose range has a compact closure, find a cluster point x of $(x_i)_{i\in\mathbb{N}}$.
- $\mathsf{COH} \equiv_{\mathrm{W}} \mathsf{WBWT}_{\mathbb{R}}$ (Kreuzer 2011)
- $\mathsf{WBWT}_{\mathbb{R}}$ is the problem: given a sequence $(x_i)_{i\in\mathbb{N}}$ whose range has a compact closure, find a sequence $(y_i)_{i\in\mathbb{N}}$ that converges to a cluster point x of $(x_i)_{i\in\mathbb{N}}$.
- Hence it suffices to show $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \lim * \mathsf{WBWT}_{\mathbb{R}}$, where $\geq_{\mathrm{W}}$ follows from the Limit Control Theorem.

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- $\mathsf{WBWT}_{\mathbb{R}}$ is the problem: given a sequence $(x_i)_{i\in\mathbb{N}}$ whose range has a compact closure, find a sequence $(y_i)_{i\in\mathbb{N}}$ that converges to a cluster point x of $(x_i)_{i\in\mathbb{N}}$.
- Hence it suffices to show $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \lim *\mathsf{WBWT}_{\mathbb{R}}$, where $\geq_{\mathrm{W}}$ follows from the Limit Control Theorem.

□

Application 2: Decomposition of König's Lemma

For $f :\subseteq X \rightrightarrows Y$ we define the jump $f' :\subseteq X' \rightrightarrows Y$ (which is the same problem with a modified input representation).

Theorem (B., Hendtlass and Kreuzer 2017)

$\mathsf{WKL}' \equiv_W \lim * \mathsf{COH}$.

Proof.

- $\mathsf{WKL}' \equiv_{sW} \mathsf{BWT}_{\mathbb{R}}$ (B., Gherardi, Marcone 2012).
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Application 3: Classification of Ramsey's Theorem

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- RT_k^n denotes the problem: given a coloring $c : [\mathbb{N}]^n \to k$, find an infinite homogenous set H for it.
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- $\mathsf{SRT}_k^{n+1} \leq_{\mathrm{W}} \mathsf{RT}_k^n * \lim$,
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Theorem (B. and Rakotoniaina 2017)

$\mathsf{RT}_k^n \leq_{\mathrm{W}} \mathsf{WKL}^{(n)}$ *and* $\widehat{\mathsf{RT}_k^n} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$.

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Application 4: Inverting Jumps

$\leq_{\mathrm{W}}^{p}$ denotes Weihrauch reducibility relativized by an oracle $p \in \mathbb{N}^{\mathbb{N}}$, i.e., the reduction function H, K can depend on p.

Theorem (B., Hölzl and Kuyper 2017)

$f' \leq_{\mathrm{W}}^{p} g' \Longrightarrow f \leq_{\mathrm{W}}^{p'} g$.

Corollary

$f' \leq_{\mathrm{W}}^{\mathrm{c}} g' \Longrightarrow f \leq_{\mathrm{W}}^{\mathrm{c}} g$ and $f' \leq_{\mathrm{sW}}^{\mathrm{c}} g' \iff f \leq_{\mathrm{sW}}^{\mathrm{c}} g$.

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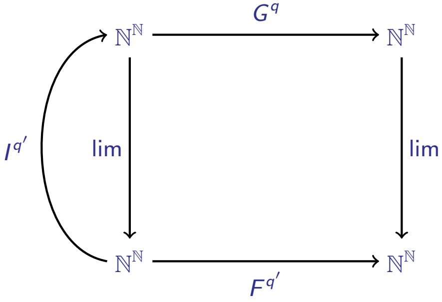

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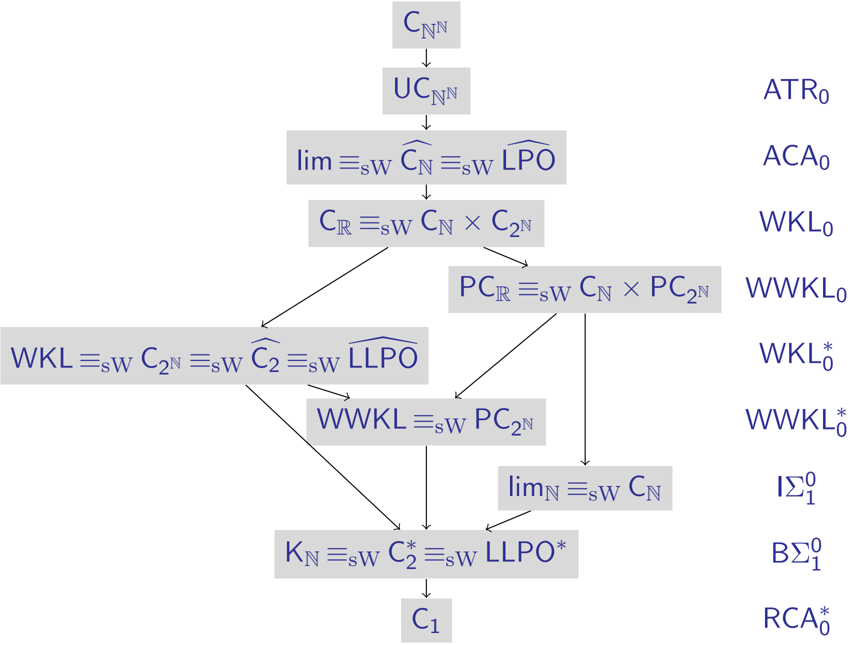
A Glimpse of the Weihrauch Lattice



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A Survey on Weihrauch Complexity

Weihrauch Complexity in Computable Analysis

Vasco Brattka, Guido Gherardi, Arno Pauly

(Submitted on 11 Jul 2017)

We provide a self-contained introduction into Weihrauch complexity and its applications to computable analysis. This includes a survey on some classification results and a discussion of the relation to other approaches.

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Abstract We provide a self-contained introduction into Weihrauch complexity and its applications to computable analysis. This includes a survey on some classification results and a discussion of the relation to other approaches.

1 The Algebra of Problems

The Weihrauch lattice offers a framework to classify the uniform computational content of problems and theorems from analysis and other areas of mathematics. This framework can be seen as an attempt to create a calculus of mathematical problems, very much in spirit of Kolmogorov's interpretation of intuitionistic logic [69].

We express mathematical problems with the help of partial multi-valued functions $f :\subseteq X \rightrightarrows Y$. We consider $\mathrm{dom}(f) = \{x \in X : f(x) \neq \emptyset\}$ as the set of admissible instances x of the problem f, and we consider the corresponding set of function values $f(x) \subseteq Y$ as the set of possible results. In the case of single-valued f we identify $f(x)$ with the corresponding singleton. An example of a mathematical problem that the reader can have in mind as a prototypical case is the zero problem. Obviously,

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