Bioinformatics
(Graph Products)

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There are four standart products:

- Cartesian product $\square$
- direct product $\times$
- strong product $\boxtimes$
- lexicographic product $\circ$

The vertex set $V(G_1 \star G_2)$, $\star \in \{\Box, \times, \boxtimes, \circ\}$

As numbers, one can multiply graphs.

The vertex set $V(G)$ of the products $\star \in \{\Box, \times, \boxtimes, \circ\}$ is defined as follows:

$$V(G_1 \star G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$$

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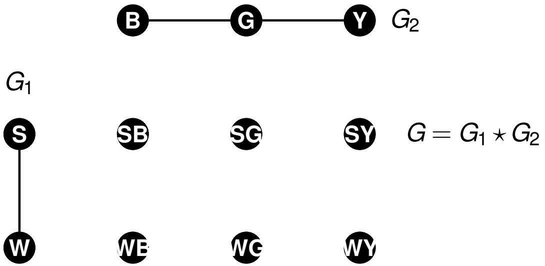
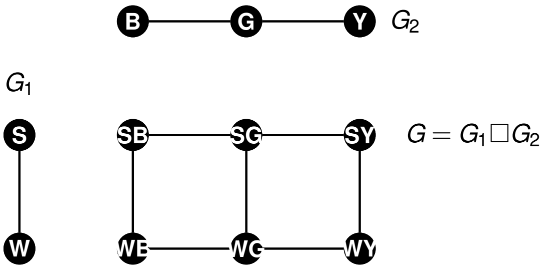


The Cartesian product $G = G_1 \square G_2$

As numbers, one can multiply graphs.

Two vertices (x_1, x_2), (y_1, y_2) in G are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ and $x_2 = y_2$ or if
2. $[x_2, y_2] \in E(G_2)$ and $x_1 = y_1$.

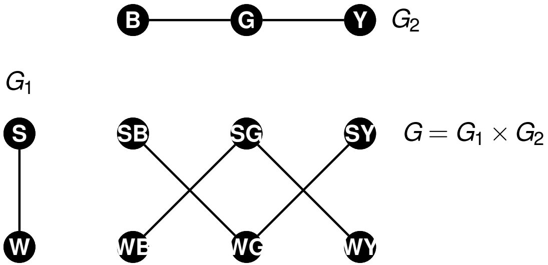


The direct product $G = G_1 \times G_2$

As numbers, one can multiply graphs.

Two vertices (x_1, x_2), (y_1, y_2) in G are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ and $[x_2, y_2] \in E(G_2)$.

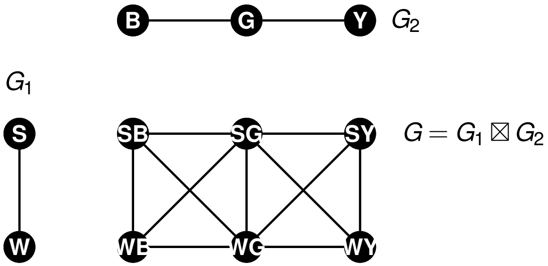


The strong product $G = G_1 \boxtimes G_2$

As numbers, one can multiply graphs.

Two vertices (x_1, x_2), (y_1, y_2) in G are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ and $x_2 = y_2$ or if
2. $[x_2, y_2] \in E(G_2)$ and $x_1 = y_1$ or if
3. $[x_1, y_1] \in E(G_1)$ and $[x_2, y_2] \in E(G_2)$.

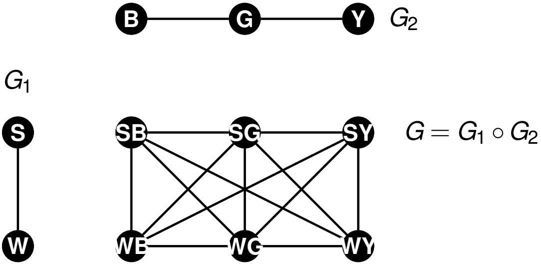


The lexicographic product $G = G_1 \circ G_2$

As numbers, one can multiply graphs.

Two vertices (x_1, x_2), (y_1, y_2) in G are linked by an edge if:

1. $[x_1, y_1] \in E(G_1)$ or if
2. $[x_2, y_2] \in E(G_2)$ and $x_1 = y_1$.



Cartesian Product: properties

- commutative
- associative
- distributive w.r.t. disjoint union $+$
- unit element K_1, i.e, for all G holds $G \Box K_1 \simeq K_1 \Box G \simeq G$.

Cartesian Product: properties

- fiber, layer
- projections (are weak homomorphisms)

Theorem

Let $G = \square_{i=1}^{n} G_i$ *and* $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V(G)$*. It holds:*

$$d_G(x, y) = \sum_{i=1}^{n} d_{G_i}(x_i, y_i).$$

Theorem

$G = \square_{i=1}^{n} G_i$ *is connected if and only if* G_i *is connected for all* $i = 1, \dots, n$*.*

Cartesian Product: properties

Lemma (Square Property)

Let $G = \Box_{i=1}^{n} G_i$ *be a Cartesian product graph and* $e, f \in E(G)$ *be two incident edges that are in different fibers. Then there is exactly one diagonalfree square in* G *containing both* e *and* f*.*

Prime Factor Decomposition (PFD) w.r.t. $\Box$

G is prime, if for all G_1, G_2 with

$$G = G_1 \Box G_2 \quad \Rightarrow \quad G_1 \simeq K_1 \text{ or } G_2 \simeq K_1$$

Theorem

Every connected graph $G = (V, E)$ has a unique representation as a Cartesian product of prime factors (up to isomorphism and the order of the factors).

The number of prime factors is at most $\log_2(|V|)$.

PFD is not unique in the class of disconnected graphs.

Prime Factor Decomposition (PFD) w.r.t. $\Box$

Aim: Find PFD of given graphs G.

Definition (Product Relation σ)

Let $G = \Box_{i=1}^{n} G_i$ be a Cartesian product graph. Two edges e, f are in relation σ, $(e \sigma f)$, if the endpoints of e, resp. f, differ exactly in the same coordinate i.

Thus, for e and f with $e\sigma f$ holds: they are edges of fibers of factor G_i. The edges e and f can than be colored with color i.

Aim: Compute "finest" σ.

Djokovic-Winkler-Relation Θ

Two edges $e = (x, y), f = (a, b)$ are in Relation Θ, $(e\Theta f)$, iff

$$d(x,a) + d(y,b) \neq d(x,b) + d(y,a)$$

Lemma

Let G be a graph. It holds:

- *For two incident edges e, f holds $e\Theta f$ if and only if e and f belong to a common triangle.*
- *Let P be a shortest path in G then no two edges of P are in Relation Θ.*
- *Let C be an isometric cycle of G. If $e, f \in E(C)$ are "antipodal" edges then $e\Theta f$.*

Θ is symmetric, reflexiv, not transitiv.

Djokovic-Winkler-Relation Θ

Two edges $e = (x, y), f = (a, b)$ are in Relation Θ, $(e\Theta f)$, iff

$$d(x,a) + d(y,b) \neq d(x,b) + d(y,a)$$

Lemma

Let e and f be edges of a Cartesian product graph G with eΘf then the endvertices of e and f differ in the same coordinate.

Thus, we can conlcude that

$$\Theta \subseteq \sigma.$$

Problem: even transitive closure Θ^* is not a product relation. Thus we consider the Relation τ.

Relation τ

Let G be a graph. Two edges $e = (u, v), f = (u, w)$ are in Relation τ, $(e\tau f)$, iff $e = f$ or $(v, w) \notin E(G)$ and u is the only common neighbor of v and w.

Theorem

The relation $(\Theta \cup \tau)^*$ is the finest product relation σ and thus corresponds to the PFD w.r.t. $\Box$ of a given graph.
$(\dots)^*$ denotes the transitive closure

PFD w.r.t. $\Box$

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INPUT: Adjacency-list of a graph G = (V, E)
Compute eqivalences F_1, ..., F_n of (Θ ∪ τ)*
for i = 1, ..., n do
    Compute an arbitrary connected component G_i of G induced by F_i
    Save G_i as prime factor
end for
OUTPUT: The prime factors G_1, ..., G_n of G
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Lemma

The PFD of $G = (V, E)$ *w.r.t. the Cartesian product can be computed in* $O(|V||E|)$ *time.*