Homotopical vector bundles on differentiable stacks

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Lie groupoids and their representations

Let $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ be a Lie groupoid. Examples:

- manifold $M \rightrightarrows M$.
- Lie group $G \rightrightarrows *$.
- action groupoid $G \ltimes M \rightrightarrows M$ for an action $G \curvearrowright M$.

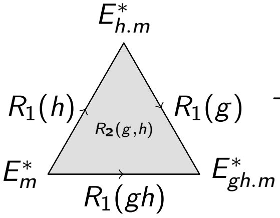
Definition

A *representation* of $\mathcal{G}$ is a vector bundle E over $\mathcal{G}_0$ and action maps $a_g : E_{s(g)} \to E_{t(g)}$ for $g \in \mathcal{G}_1$. + associativity + unitality

- vector bundle.
- representation of the Lie group.
- equivariant vector bundle.

Representations up to homotopy

$G \ltimes M \rightrightarrows M$ action groupoid. A *representation up to homotopy* is:

A dg vector bundle (E^*, ∂) over M + 

+ Higher coherent homotopies R_n

quasi-action

$\rightsquigarrow$ dg-category $\mathrm{Rep}_\infty(\mathcal{G})$

Conjecture ([AC09])

The functor $\mathrm{Rep}_\infty : \mathrm{Grpd}^{\mathrm{op}} \to \mathrm{dg}\mathcal{C}\mathrm{at}$ *maps Morita equivalences to weak equivalences. Equivalently,* Rep_∞ *factors through* Grpd $\to$ Stacks.

Homotopy Kan extension to stacks

Start with $\mathcal{F} : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathcal{C}$. Write:

$$[\mathcal{G}_0//\mathcal{G}_1] = \mathsf{hocolim}_{\Delta^{\mathrm{op}}}(\mathcal{G}_0 \leftleftarrows \mathcal{G}_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathcal{G}_2 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \dots)$$

Define

$$\mathcal{F}([\mathcal{G}_0//\mathcal{G}_1]) := \mathsf{holim}_{\Delta}\, \mathcal{F}(\mathcal{G}_n)\,.$$

This is a right Kan extension to (pre)stacks.

$$\begin{array}{ccc} \mathcal{M}\mathrm{fld}^{\mathrm{op}} & \xrightarrow{\mathcal{F}} & \mathcal{C} \\ \big\downarrow & \nearrow & \\ \mathsf{(Pre)Stacks}^{\mathrm{op}} & & \end{array}$$

e.g. $\mathcal{F} = C^{\infty} : \mathcal{M}\mathrm{fld}^{\mathrm{op}} \to \mathsf{Ch}_{\mathbb{R}}$.

Extending $\mathrm{dg}\mathcal{V}\mathrm{ec}$ to stacks

Consider the functor

$$\begin{aligned} \mathrm{dg}\mathcal{V}\mathrm{ec} : \mathcal{M}\mathrm{fld}^{\mathrm{op}} &\longrightarrow \mathrm{dg}\mathcal{C}\mathrm{at} \\ M &\longmapsto \mathrm{dg}\mathcal{V}\mathrm{ec}(M)\,. \end{aligned}$$

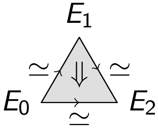
Extend this functor:

$$\mathrm{dg}\mathcal{V}\mathrm{ec}([\mathcal{G}_0//\mathcal{G}_1]) := \mathsf{holim}_\Delta\, \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n)$$

Use Bousfield-Kan formula:

$$\mathsf{holim}_\Delta\, \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n) \simeq \int_\Delta \underbrace{\mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n)^{\Delta^n}}_{\text{need to make sense of this}}$$

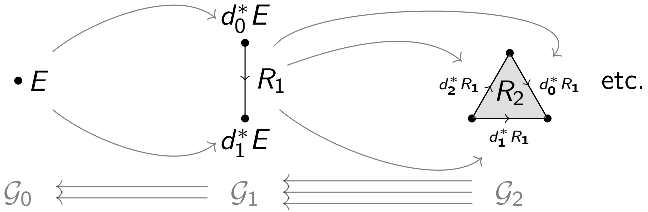
$$\operatorname{holim}_\Delta \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n) \simeq \int_\Delta \underbrace{\mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_n)^{\Delta^n}}_{\text{c.f. [AO19]}}$$



$$\in \mathrm{dg}\mathcal{V}\mathrm{ec}(\mathcal{G}_2)^{\Delta^2}$$

Theorem (DA)

The dg categories $\mathrm{dg}\mathcal{V}\mathrm{ec}([\mathcal{G}_0//\mathcal{G}_1])$ *and* $\mathrm{Rep}_\infty(\mathcal{G})$ *are weakly equivalent.*



An approach to the Morita invariance of Rep_∞

- We saw that we can basically identify Rep_∞ and $\mathrm{dg}\mathcal{V}\mathrm{ec}$.
- $\mathrm{dg}\mathcal{V}\mathrm{ec}$ *does not* satisfy descent, i.e. we cannot glue dg vector bundles. But it is reasonably close to satisfying descent.
- *Suppose* it did satisfy descent. Then, it would factor through the stackification functor:

$$\begin{array}{ccc} \text{PreStacks} & \xrightarrow{\mathrm{dg}\mathcal{V}\mathrm{ec}} & \mathrm{dg}\mathcal{C}\mathrm{at} \\ \downarrow & \nearrow & \\ \text{Stacks} & & \end{array}$$

  In particular, it would invert Morita equivalences.
- We have thus removed the difficulty of dealing with stacks to a descent problem about a functor on manifolds.

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Thank you for your attention!

IMPRS Moduli Spaces

Camilo Arias Abad and Marius Crainic.
Representations up to homotopy and Bott's spectral sequence for Lie groupoids, 2009.

Sergey Arkhipov and Sebastian Oersted.
Homotopy limits in the category of dg-categories in terms of A_∞-comodules, 2019.