Units of equivariant K-theory and bundles of C^*-algebras

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Classical Dixmier-Douady theory

Let X be a compact space and let $\mathbb{K}$ denote the compact operators on a separable Hilbert space.

- There is a natural isomorphism

$$\delta : \{\text{Bundles over } X \text{ with fiber } \mathbb{K}\} \to [X, B\mathrm{Aut}(\mathbb{K})] \cong H^3(X, \mathbb{Z})$$

  which is multiplicative, in the sense that

$$\delta(\mathcal{K}_1 \otimes \mathcal{K}_2) = \delta(\mathcal{K}_1) + \delta(\mathcal{K}_2)$$

- The multiplicative structure comes from the isomorphism $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$
- Can this be generalised?

Strongly self-absorbing C^*-algebras

Definition

A separable, unital C^*-algebra D is called strongly self-absorbing if $\exists$ an isomorphism $\psi : D \to D \otimes D$ and a continuous path $u : [0,1) \to U(D \otimes D)$ with

$$\lim_{t \to 1} \|\psi(d) - u_t(d \otimes 1_D)u_t^*\| = 0.$$

Examples: $\mathbb{C}, \mathrm{End}(V)^{\otimes \infty}, \mathcal{O}_\infty, \mathcal{O}_2, \mathcal{Z}$

Topological properties:

- $\mathrm{Aut}(D)$ is contractible
- $K_0(D)$ is a ring
- $X \mapsto K_*(C_0(X) \otimes D)$ defines a multiplicative cohomology theory. Denote by KU^D the corresponding spectrum (for $D = \mathbb{C}$ we get $KU^{\mathbb{C}} = KU$).

Units of KU^D

Definition

The space of units $\mathrm{GL}_1(KU^D)$ is defined by the following pullback:

$$\begin{array}{ccc} \mathrm{GL}_1(KU^D) & \longrightarrow & \Omega KU_1^D \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{GL}_1(K_0(D)) & \longrightarrow & K_0(D) \end{array}$$

- The spectrum KU^D also has a multiplicative structure coming from the tensor product, making it into a (commutative) ring spectrum
- It turns out that $\mathrm{GL}_1(KU^D)$ is actually the 0^{th}-space of a spectrum (called the spectrum of units $gl_1(KU^D)$).

The homotopy type of Aut$(D \otimes \mathbb{K})$

Theorem (Dadarlat-Pennig)

- Aut$(D \otimes \mathbb{K})$ is an infinite loop space
- its homotopy groups are

$$\pi_i(\mathrm{Aut}(D \otimes \mathbb{K})) = \begin{cases} \mathrm{GL}_1(K_0(D))_+ & \text{if } i = 0 \\ K_i(D) & \text{if } i \geq 1 \end{cases}$$

- it defines a cohomology theory E_D with

$$E_D^0(X) = [X, \mathrm{Aut}(D \otimes \mathbb{K})] \text{ and } E_D^1(X) = [X, B\mathrm{Aut}(D \otimes \mathbb{K})]$$

- and the group law coming from the infinite loop space structure coincides with the $\otimes$ of $D \otimes \mathbb{K}$-bundles

Is E_D a known cohomology theory?

E_D is the spectrum of positive units of KU^D-theory

Theorem (Dadarlat-Pennig)

There is a weak homotopy equivalence

$$\mathrm{Aut}(D \otimes \mathbb{K}) \to \mathrm{GL}_1(KU^D)_+$$

which lifts to a homotopy equivalence of the corresponding spectra.

- Higher Dixmier-Douady theory:

  $$\{\text{Bundles}/X \text{ with fiber } D \otimes \mathbb{K}\} \cong [X, BGL_1(KU^D)_+] \cong gl_1(KU^D)^1_+(X)$$

- Operator-algebraic description of higher twists of K-theory

Equivariant bundles

- Let G be a finite group and H be a countable discrete group
- Each continuous action $\alpha : G \times H \to \mathrm{Aut}(D \otimes \mathbb{K})$ gives rise to a G-equivariant bundle

$$\mathcal{A} = (E(G \times H) \times_{\alpha|_H} (D \otimes \mathbb{K}))/\sim \to B_G H$$

  over the equivariant classifying space $B_G H$
- Such equivariant bundles are classified by the set

$$[B_G H, B\mathrm{Aut}(D \otimes \mathbb{K})]^G$$

- Meyer, Gabe and Szabó: if D is a Kirchberg algebra in the UCT-class, G trivial and H amenable and torsion-free then

$$\{\text{Out. act. of } H \text{ on } D \otimes \mathbb{K}\}/\sim_{cc} \cong [BH, B\mathrm{Aut}(D \otimes \mathbb{K})] \underset{BH \text{ finite } CW}{\cong} E^1_D(BH)$$

The equivariant case - First steps

- Motivated by constructions in equivariant twisted K-theory, Evans and Pennig studied the case of $G = S^1$ and $D = \mathrm{End}(V)^{\otimes\infty}$ for V a S^1-representation
- They found that $\mathrm{Aut}_{S^1}(D \otimes \mathbb{K})$ is still an infinite loop space, and its homotopy groups in positive even degrees are proper subgroups of $K_i^G(D)$
- For $G = \mathbb{Z}/p\mathbb{Z}$ (restricted action on $D = \mathrm{End}(V)^{\otimes\infty}$) we have

$$\pi_i(\mathrm{Aut}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K})) = \begin{cases} \mathrm{GL}_1(K_0^{\mathbb{Z}/p\mathbb{Z}}(D))_+ & \text{if } i = 0 \\ K_i^{\mathbb{Z}/p\mathbb{Z}}(D) & \text{if } i \geq 1 \end{cases}$$

- Can we repeat the story equivariantly?

Equivariant units of KU^D

- Assume the ring spectrum KU^D has a G-action (in our case, a $\mathbb{Z}/p\mathbb{Z}$-action induced by the one on $D \otimes \mathbb{K}$)
- We define the space of equivariant units $\mathrm{GL}_1(KU^D)$ in such a way that $(GL_1(KU^D))^H$ realises the following pullback for any $H \leq G$:

$$\begin{array}{ccc} (\mathrm{GL}_1(KU^D))^H & \longrightarrow & (\Omega KU^D_1)^H \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{GL}_1(K_0^H(D)) & \longrightarrow & K_0^H(D) \end{array}$$

- We show that also in the equivariant case (for a finite group G) $\mathrm{GL}_1(KU^D)$ is the 0^{th}-space of a G-spectrum of units $gl_1(KU^D)$.

The $\mathbb{Z}/p\mathbb{Z}$-equivariant case

Theorem (B.-Pennig)

Let $G = \mathbb{Z}/p\mathbb{Z}$ and $D = \mathrm{End}(V)^{\otimes\infty}$. The G-spectrum coming from $\mathrm{Aut}(D \otimes \mathbb{K})$ is homotopy equivalent to $gl_1(KU^D)_+$.

Corollary

$\mathrm{Aut}(D \otimes \mathbb{K})$ is an infinite loop space with associated $\mathbb{Z}/p\mathbb{Z}$-equivariant cohomology theory $E^*_D(X) = gl_1(KU^D)_+(X)$, and

$$E^0_{D,H}(X) = [X, \mathrm{Aut}(D \otimes \mathbb{K})]^H \quad \text{and} \quad E^1_{D,H}(X) = [X, B\mathrm{Aut}(D \otimes \mathbb{K})]^H$$

for $H \leq \mathbb{Z}/p\mathbb{Z}$. In particular, isomorphism classes of $\mathbb{Z}/p\mathbb{Z}$-equivariant C^*-algebra bundles with fibres isomorphic to the $\mathbb{Z}/p\mathbb{Z}$-algebra $D \otimes \mathbb{K}$ over the finite CW-complex X form a group with respect to the fibrewise tensor product that is isomorphic to $E^1_{D,\mathbb{Z}/p\mathbb{Z}}(X)$.

What is next?

The case $G = \mathbb{Z}/p\mathbb{Z}$ (upcoming preprint joint with U.Pennig) is only a starting point. The following are some ideas for future work:

- Can any of this be generalised to other group actions on strongly self-absorbing C^*-algebras?
  A first step would be the case $D = \mathrm{End}(V)^{\otimes\infty}$ and G a finite group.
- Is there an interpretation of the results for the case $G = S^1$ in terms of stable homotopy theory?

Thank you for your attention!