The Spectral Model for (Real) K-theory

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May 16, 2024

Introduction

Classically, operator K-theory has involved a lot of technical play with projections and matrix algebras.

Fundamental properties like stability and long exact sequences required a lot of book-keeping.

The real picture is hardly discussed, and also one has to be careful while handling non-unital C*-algebras.

Introduction

Classically, operator K-theory has involved a lot of technical play with projections and matrix algebras.

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The real picture is hardly discussed, and also one has to be careful while handling non-unital C*-algebras.

Higson and Guentner came up with a model for operator K-theory which takes care of these difficulties.

The Main theorem

Theorem

There is a natural isomorphism

$$\mathcal{F}_A : \pi_0(\mathbb{K}(A)) \to K_0^{\mathrm{Fr}}(A)$$

for unital (graded, Real) C^*-algebras.

Furthermore, when A is ungraded, there is a natural isomorphism between $K_0^{\mathrm{Fr}}(A)$ and $K_0(A)$.

Definitions and examples

Definition

A $\mathbb{Z}/2\mathbb{Z}$-graded Real C^*-algebra (A, ε, τ) is a complex C^*-algebra A equipped with a grading ε and a Real structure τ which commute, that is, $\varepsilon \circ \tau = \tau \circ \varepsilon$.

Definitions and examples

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Example

$C_0(\mathbb{R})$ becomes a $\mathbb{Z}/2\mathbb{Z}$-graded Real C^*-algebra when equipped with the even-odd grading, and the Real structure given by point-wise conjugation. We denote this $\mathbb{Z}/2\mathbb{Z}$-graded Real C^*-algebra by $\mathcal{S}$.

Definition

Let A be $\mathbb{Z}/2\mathbb{Z}$-graded (Real) C^*-algebra, and $\mathcal{H}$ be a graded (Real) Hilbert space. Define *the spectral K-theory space of A underlying $\mathcal{H}$* to be the space $\mathbb{K}(A, \mathcal{H}) := \mathrm{Hom}_{C^*}(\mathcal{S}, A\hat{\otimes}\mathcal{K}(\mathcal{H}))$, equipped with the point-norm topology.

Remark

Without the mention of the variable Hilbert space, we take it to be $\hat{l^2}$. We also don't mention the Real structure everywhere, it is assumed to be there (the reader can always complexify by ignoring it)

Definitions and examples

Example

Let D be an odd, self-adjoint operator on $\hat{l^2}$ with compact resolvents. Then, the functional calculus of D, $\Phi_D : \mathcal{S} \to \mathcal{K}(\hat{l^2})$ is an element of $\mathbb{K}(\mathbf{C})$. Moreover, if P_0 denotes the projection onto the kernel of D, then Φ_D and the graded homomorphism $f \mapsto f(0)P_0$ lie in the same path-component of $\mathbb{K}(\mathbf{C})$.

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Example

Let A be a graded C*-algebra and let F be an odd, self-adjoint essentially unitary operator on $\hat{\mathcal{H}_A}$. Then, the functional calculus of F,

$$\Phi_F : C_0(-1,1) \to \mathcal{K}(\hat{\mathcal{H}_A}) \cong A \hat{\otimes} \mathcal{K}(\hat{l^2})$$

gives an element of $\mathbb{K}^{\mathrm{bd}}(A)$.

Definitions and examples

Definition

Given a $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebra A, we define its spectral K-theory groups to be the homotopy groups of $\mathbb{K}(A)$ at the basepoint $\underline{0}$. That is, we define ${K_i}^{sp}(A) := \pi_i(\mathbb{K}(A), \underline{0})$.

Fredholm picture of K-theory

Definition

A Fredholm cycle over A is a triple $(\mathcal{E}, \iota, F)$, where $(\mathcal{E}, \iota)$ is a countably generated $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert A- module and F is an odd, self-adjoint Fredholm operator on it.

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Definition

Two Fredholm cycles $(\mathcal{E}_0, \iota_0, F_0)$ and $(\mathcal{E}_1, \iota_1, F_1)$ are said to be concordant (denoted $(\mathcal{E}_0, \iota_0, F_0) \sim (\mathcal{E}_1, \iota_1, F_1)$) if there exists $(\mathcal{E}, \iota, F)$, a Fredholm cycle over $A \hat{\otimes} C[0,1]$ such that $(\mathcal{E} \hat{\otimes}_{\mathrm{ev}_i} A, \mathrm{ev}_i(\iota)\, \mathrm{ev}_i(F)) \cong (\mathcal{E}_i, \iota_i, F_i)$ for $i = 0, 1$, where $\mathrm{ev}_t : A[0,1] \to A$ is the graded homomorphism given by $\mathrm{ev}_t(f) = f(t)$.

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Definition

Denote by $K_0^{\mathrm{Fr}}(A)$ the abelian monoid of concordance classes of Fredholm cycles over A.

Fredholm picture of K-theory

Example

Let A be a unital and ungraded C*-algebra, and let $\mathcal{E}, \mathcal{F}$ be finitely generated projective A-modules. Then, $(\mathcal{E} \oplus \mathcal{F}, \tau_{\mathcal{E},\mathcal{F}}, 0)$ is a Fredholm cycle over A.

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Example

Let A be a graded C^*-algebra, and $(\mathcal{E}, \iota)$ a graded countably generated Hilbert module over A. Let $\varphi : C_0(-1,1) \to \mathcal{K}(\mathcal{E})$ be a **non-degenerate** $*$-homomorphism. Then, φ extends to a unital $*$-homomorphism $\tilde{\varphi} : C[-1,1] \to \mathcal{B}(\mathcal{E})$, and $(\mathcal{E}, \iota, \tilde{\varphi}(x))$ is a Fredholm cycle over A

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Definition of $\mathcal{F}_A$:

$$\mathbb{K}^{\mathrm{bd}}(A) \ni \varphi \longrightarrow (\overline{\varphi(C_0(-1,1))\hat{\mathcal{H}}_A}, \tilde{\varphi}(x)) \in K_0^{\mathrm{eu}}(A)$$

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Besides injectivity and surjectivity, well-definedness needs to checked too!

Thank You!