

E-theory is compactly assembled

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E -theory is a functor $e : C^* \mathbf{Alg}^{\text{nu}} \rightarrow E$ with E in Pr_{st}^L satisfying $h, \mathcal{K}, \text{ex}, \aleph_1$, such that

$$\text{Fun}^L(E, \mathcal{C}) \xrightarrow[e^*]{\cong} \text{Fun}^{h, \mathcal{K}, \text{ex}, \aleph_1}(C^* \mathbf{Alg}^{\text{nu}}, \mathcal{C}),$$

for all \mathcal{C} in Pr_{st}^L .

Remark

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- $\text{Ind}_{\aleph_1}(C^* \mathbf{Alg}_{\text{sep}}^{\text{nu}}) \simeq C^* \mathbf{Alg}^{\text{nu}}$ implies

$$\text{Fun}^{h, \mathcal{K}, \text{ex}, \aleph_1}(C^* \mathbf{Alg}^{\text{nu}}, \mathcal{C}) \simeq \text{Fun}^{h, \mathcal{K}, \text{ex}}(C^* \mathbf{Alg}_{\text{sep}}^{\text{nu}}, \mathcal{C}).$$

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Definition

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\mathcal{C} in Pr^L : \mathcal{C} is compactly assembled $\Leftrightarrow \mathcal{C}$ is a retract of a compactly generated category in Pr^L

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- $\mathcal{C} \rightarrow \mathcal{D}$ BL with \aleph_0 -accessible right adjoints: If \mathcal{C} is cas $\Rightarrow \mathcal{D}$ is cas

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- There exists

$$\text{Ind}(C^* \mathbf{Alg}_{\text{sep,h}}^{\text{nu}}) \rightarrow \bullet \rightarrow \bullet \rightarrow E,$$

a sequence of BL with accessible right adjoints.

Why?

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$$\begin{array}{ccc} \text{Cat}^{\text{perf}} & \xrightarrow{K} & \text{Sp} \\ \downarrow & \nearrow \exists! & \\ \text{Pr}_{\text{st}}^{\text{cas}} & \xrightarrow{K} & \end{array}$$

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The diagram shows a commutative triangle. A solid arrow labeled K points from Cat^{perf} to Sp . A solid arrow points from Cat^{perf} down to $\text{Pr}_{\text{st}}^{\text{cas}}$. A dashed arrow labeled K points from $\text{Pr}_{\text{st}}^{\text{cas}}$ up to Sp . A dashed arrow labeled $\exists!$ points from $\text{Pr}_{\text{st}}^{\text{cas}}$ to the dashed arrow K , indicating a unique factorization property.

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- $K(E)$
- $K(KU) \rightarrow K(\text{Mod}(KU)) \rightarrow K(E)$