E-theory is compactly assembled

Ulrich Bunke   Benjamin Dünzinger

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E-theory is a functor $e : C^*\mathbf{Alg}^{\mathrm{nu}} \to E$ with E in $\mathrm{Pr}^L_{\mathrm{st}}$ satisfying $h, \mathcal{K}, \mathrm{ex}, \aleph_1$, such that

$$\mathrm{Fun}^L(E, \mathcal{C}) \underset{e^*}{\xrightarrow{\simeq}} \mathrm{Fun}^{h,\mathcal{K},\mathrm{ex},\aleph_1}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathcal{C}),$$

for all $\mathcal{C}$ in $\mathrm{Pr}^L_{\mathrm{st}}$.

Remark

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- $\mathrm{Ind}_{\aleph_1}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}) \simeq C^*\mathbf{Alg}^{\mathrm{nu}}$ implies

$$\mathrm{Fun}^{h,\mathcal{K},\mathrm{ex},\aleph_1}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathcal{C}) \simeq \mathrm{Fun}^{h,\mathcal{K},\mathrm{ex}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathcal{C}).$$

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$\mathcal{C}$ in Pr^L: $\mathcal{C}$ is compactly assembled $\Leftrightarrow$ $\mathcal{C}$ is a retract of a compactly generated category in Pr^L

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- $\mathcal{C} \to \mathcal{D}$ BL with $\aleph_0$-accessible right adjoints: If $\mathcal{C}$ is cas $\Rightarrow$ $\mathcal{D}$ is cas

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- There exists

$$\mathrm{Ind}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep,h}}) \to \bullet \to \bullet \to E,$$

  a sequence of BL with accessible right adjoints.

Why?

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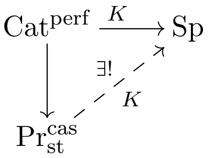
- Efimov:



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$$\begin{array}{ccc} \mathrm{Cat}^{\mathrm{perf}} & \xrightarrow{K} & \mathrm{Sp} \\ \downarrow & \overset{\exists !}{\nearrow}{\scriptstyle K} & \\ \mathrm{Pr}^{\mathrm{cas}}_{\mathrm{st}} & & \end{array}$$

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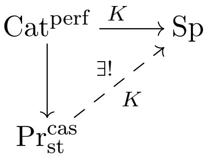

- $K(E)$
- $K(KU) \to K(\mathrm{Mod}(\mathrm{KU})) \to K(E)$