# Coarse Geometry

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From Analysis to Homotopy Theory
A conference in honor of Ulrich Bunke's 60th birthday

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# Trivia about the project

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- A (conservative) count finds 19 papers on the arXiv totalling 1367 pages and 7 different co-authors[1].
- Quote from the first notes (dating back to November 2015): *There is no interesting homotopy theory on coarse spaces.*

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  - Comparison of coarse homology theories.
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  - Hochschild and cyclic homology.
- Non-trivial results:
  - Comparison of coarse homology theories.
  - Construction of coarse assembly maps and corresponding isomorphism results.
- Equivariant theory gives results on the Baum–Connes and Farrell–Jones conjectures.

# Coarse structures

## Definition

A coarse structure on the set $X$ is a collection $\mathscr{C}$ of subsets of $X \times X$ (called entourages) such that:

- The diagonal of $X$ is an entourage.
- $\mathscr{C}$ is closed under finite unions and subsets.
- $\mathscr{C}$ is closed under flipping

$$U \mapsto U^{-1} := \{(y, x) \colon (x, y) \in U\}$$

  and composition

$$U \circ V := \{(x, y) \colon \exists z \in X \text{ with } (x, z) \in U \text{ and } (z, y) \in V\}.$$

# Metric coarse structure

## Example

Let $(X, d)$ be a metric space. Its canonically associated coarse structure is the one generated by the entourages

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for all $r \in [0, \infty)$.

**Remark**

*I think about a coarse space $(X, \mathscr{C})$ in the following way:*

*If $E \in \mathscr{C}$ is an entourage, then $(x, y) \in E$ means to me that „the distance from $x$ to $y$ is at most $E$."*

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## Definition

*A map $f\colon X \to Y$ between coarse spaces is*

- *controlled, if $f \times f$ maps entourages to entourages,*
- *proper, if preimages under $f$ of bounded sets are bounded.*

*We call $f$ a coarse map, if it is controlled and proper.*

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The appropriate notion of „isomorphisms“ in coarse geometry is the following one:

**Definition**

$f, g\colon X \to Y$ are close to each other, if $\{(f(x), g(x))\colon x \in X\}$ is an entourage of $Y$.

$f\colon X \to Y$ is a coarse equivalence, if there exists an $h\colon Y \to X$ such that $f \circ h$ and $h \circ f$ are close to the respective identity maps.

# Main example

## Example

Let $M$ be, say, a closed manifold (equipped with any Riemannian metric) and denote by $X$ its universal cover. Then, up to coarse equivalence, we have the following:

- $X$ can be discretized.
  Concretely, it is coarsely equivalent to $\pi_1(M)$.
- $X$ can (often) be made contractible.
  Concretely, it is coarsely equivalent to $E\pi_1(M)$.[a]

[a] Provided $B\pi_1(M)$ has a finite model.

# Axioms for coarse homology theories

Let **M** be a cocomplete stable $\infty$-category and $E\colon \mathbf{Coarse} \to \mathbf{M}$ be a functor.

**Definition**

$E$ is a coarse homology theory provided

- $E$ is coarsely invariant,
- $E$ is excisive,
- $E$ annihilates flasque spaces,
- $E$ is $u$-continuous.

# Flasque spaces

## Definition

A coarse space $X$ is flasque, if it admits a self-map $f\colon X \to X$ such that

- $f$ is close to $\mathrm{id}_X$,
- $f$ is non-expanding, i.e. for every entourage $U$ of $X$ the union $\bigcup_{n\in\mathbb{N}}(f^n \times f^n)(U)$ is again an entourage,
- $f$ shifts $X$ to infinity, i.e. for every bounded set $B \subset X$ exists an $n \in \mathbb{N}$ such that $B \cap f^n(X) = \emptyset$.

## Example

The ray $[0, \infty)$ is flasque.

# $u$-Continuity

If $U$ is any entourage of a coarse space $X$, we can consider the new coarse space $X_U$ with a (possibly) smaller coarse structure: The one generated just by $U$.

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My interpretation for this is the following: Every cycle of a coarse homology theory has a „propagation“.

# Coarsification via Rips complexes

Let $U$ be an entourage of $X$. Define the Rips complex $P_U(X)$ as the simplicial complex whose simplices are exactly those whose vertices are at distance at most $U$ from each other.

We have inclusions of subcomplexes $P_U(X) \to P_V(X)$ if $U \subset V$.

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**Definition**

If $E^{\mathrm{top}}$ is a locally finite homology theory, define its coarsification

$$E^{\mathrm{coarse}}(X) := \mathrm{colim}_U\, E^{\mathrm{top}}(P_U(X)).$$

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**Definition**

If $E^{\mathrm{top}}$ is a locally finite homology theory, define its coarsification

$$E^{\mathrm{coarse}}(X) := \mathrm{colim}_U\, E^{\mathrm{top}}(P_U(X)).$$

Nota that if $X$ is (suitably) contractible, then $P_U(X)$ will be properly homotopy equivalent to $X$ and hence $E^{\mathrm{coarse}}(X) \cong E^{\mathrm{top}}(X)$.

# Cone functors

Let $(X, d)$ be a metric space. Define the cone $\mathcal{O}(X)$ as $[1, \infty) \times X$ with the metric given by $t \cdot d$ on a slice $\{t\} \times X$.[2]

[2] This is an ad hoc definition for this talk. The actual def'n is more technical.

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Let $(X, d)$ be a metric space. Define the cone $\mathcal{O}(X)$ as $[1, \infty) \times X$ with the metric given by $t \cdot d$ on a slice $\{t\} \times X$.[2]

If $E^{\mathrm{coarse}}$ is a coarse homology theory, define the corresponding topological theory by

$$E^{\mathrm{top}}(X) := E^{\mathrm{coarse}}(\mathcal{O}(X), X).$$

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# Coarse assembly

Give a coarse homology theory $E$, we get a new coarse homology theory $E^{\mathrm{as}}$ by first turning $E$ into a topological theory and then back into a coarse one again, i.e.

$$E^{\mathrm{as}}(X) = \operatorname{colim}_U E(\mathcal{O}(P_U(X)), P_U(X)).$$

This comes with a natural transformation $E^{\mathrm{as}} \to \Sigma E$, the coarse assembly map.

# Wrong way maps

**Theorem (E. 2017)**

*Let $M$ be a closed manifold and $N \hookrightarrow M$ a submanifold of codimension $q$. Assume that the normal bundle of $N$ is oriented, that $\Lambda := \pi_1(N)$ injects in $G := \pi_1(M)$, and $\pi_i(M) = 0$ for $2 \leq i \leq q$.*

*Then there exists a „nice“ map $H_*(BG) \to H_{*-q}(B\Lambda)$.*

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**Proof.**

$$\begin{array}{ccccccc} H_*(BG) & & & & & & H_{*-q}(B\Lambda) \\ \downarrow \cong & & & & & & \downarrow \cong \\ H\mathcal{X}_*^G(EG) & \xleftarrow{\cong} & H\mathcal{X}_*^G(\tilde{M}) & \dashrightarrow & H\mathcal{X}_{*-q}^\Lambda(\tilde{N}) & \xrightarrow{\cong} & H\mathcal{X}_{*-q}^\Lambda(E\Lambda) \end{array}$$

Thanks for your attention and

# Happy birthday Uli!