Higher Gauge Theory, Definition and Classifying Space

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A Little Story

- Principal ∞-bundles are defined (1207.0248, 2308.04196).
- The notion of connection is more subtle to define and existing definitions suffer from FAKE FLATNESS.
- Physists have constructed many non-trivial examples of higher gauge theory, specially, in the context of M-theory, heterotic supergravity and gauged supergravities (see the survey 2401.05275 for encyclopedia of Mathematical Physics).

This Talk

- In this talk, I reformulate the notion of connection on principal bundles and only state the result for other cases.
- I will work in the cocycle description. In particular, this leads to Giraud's non-abelian cohomology topologically.

Principal Bundles' Classifying Space

Given a Lie group G, the corresponding classifying space $\mathrm{B}G$ is the Lie groupoid $G \rightrightarrows \star$. In particular, any principal G-bunlde over a smooth manifold X is given by an open cover $\sigma = \{U_i\}$ and a functor, $F : \check{\mathscr{C}}(\sigma) \to \mathrm{B}G$, from the corresponding Čech groupoid to the classifying space. Taking limit on the category of open covers gives you the Giraud's non-abelian cohomology.

Severa Differentiation

Severa Differentiation

Given a Lie group G, the corresponding dg-manifld $\mathfrak{g}$ is defined as the representation of the presheaf
$\mathsf{Hom}(\mathsf{i}(-) \times \mathscr{P}\mathsf{air}(\Theta), \mathsf{B}G) : \mathsf{NQMfds} \to \mathsf{Sets}$, (arXiv:0612349).

In the defintion above, the map $\mathsf{i} : \mathsf{NQMfds} \to \mathsf{NQGrps}$ considers an NQ-manifold trivially as a NQ-groupoid, Θ is the shifted real line.

Local Connection

Locally a connection is a dg-map $\mathrm{T}[1]X \to \mathrm{T}[1]\mathfrak{g}$. A dg-map $\mathrm{T}[1]X \to \mathfrak{g}$ is a flat connection.

Extended Severa Differentiation

We lift the definition of differentiation to inner hom.

Extend Severa Differentiation

Given a a Lie group G, the corresponding dg-Lie groupoid $\mathscr{A}(G)$ is defined as the representation of the 2-presheaf $\mathsf{Hom}(- \times \mathscr{P}\mathsf{air}(\Theta), \mathsf{B}G) : \mathsf{NQGrps} \to \mathsf{Grps}$. Obviously $\mathsf{Ob}(\mathscr{A}(G)) = \mathfrak{g}$.

Indeed, we calculate that $\mathscr{A}(G)$ is

$$\begin{array}{c} \mathfrak{g} \times T[1]G \\ \downarrow\downarrow \\ \mathfrak{g} \end{array}$$

Principal bundles with Flat Connection

Theorem

Principal G-bundles with flat connections are classified by $\mathscr{A}(G)$. In other words $\mathrm{B}\mathscr{G}_f^{con} = \mathscr{A}(G)$.

More precisely, Principal G-bundles with flat connections are classified with an open cover σ and a dg-functor $\omega : T[1]\check{\mathscr{C}}(\sigma) \to \mathscr{A}(G)$, gauge transformations are also dg-natural transformations.

Principal bundles with Connection

definition

The classifying space $\mathrm{B}G^{con}$ is a dg Lie groupod such that

- $\mathrm{Ob}(\mathrm{B}G^{con}) = \mathrm{T}[1]\mathfrak{g}$.
- there is a dg functor $F : \mathscr{A}(G) \to \mathrm{B}G^{con}$.
- the dg-map $F_0 : \mathrm{Ob}(\mathscr{A}(G)) = \mathfrak{g} \to \mathrm{Ob}(\mathrm{B}G^{con}) = \mathrm{T}[1]\mathfrak{g}$ is the canonical one.
- the diagram

$$\begin{array}{ccc} \mathrm{Mor}(\mathscr{A}(G)) & \xrightarrow{F_1} & \mathrm{Mor}(\mathrm{B}G^{con}) \\ \downarrow{\scriptstyle \mathrm{t}} & & \downarrow{\scriptstyle \mathrm{t}} \\ \mathfrak{g} & \xrightarrow{F_0} & \mathrm{T}[1]\mathfrak{g} \end{array}$$

is a pullback diagram.

Conclisions

Theorem

The dg-Lie groupoid $\mathrm{B}G^{con}$ exists and is unique. In paricular, it is given by

$$\begin{array}{c} T[1]\mathfrak{g} \times T[1]G \\ \downdownarrows \\ T[1]\mathfrak{g} \end{array}$$

Conclisions

- In the case of Lie 2-groups, One can construct examples using adjustment datum which is used in the literature to construct non-abelian gauge theory avoiding fake flatness (arXiv: 2203.00092).
- In the case of Lie groupoids, One can construct examples using Cartan connections on Lie groupoids.

Thanks for your attention.