Some Lax Constructions in Higher Category Thy

Notation: $n\mathrm{Cat} = \mathrm{Cat}_{(\infty,n)}$

Example: $X \in n\mathrm{Cat}$, consider $\Sigma_1 X \in (n+1)\mathrm{Cat}$

Here $\Sigma_1 X = \{ 0 \xrightarrow{X} 1 \}$, i.e.

$$\mathrm{Map}_{\Sigma_1 X}(i,j) = \begin{cases} * & i=j \\ X & i<j \\ \emptyset & i>j \end{cases}.$$

$$\Sigma_1 : n\mathrm{Cat} \longrightarrow (n+1)\mathrm{Cat}_{\{0,1\}/}$$

Note: $\{0,1\} = \Sigma_1 \emptyset \longrightarrow \Sigma_1 X$.

Σ_1 has a right adjoint $\Omega : (n+1)\mathrm{Cat}_{\{0,1\}/} \to n\mathrm{Cat}$

where

$$\begin{tikzcd} \Omega X = \mathrm{Map}_X(0,1) \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \{0\} \arrow[r] \arrow[ur, phantom, "\Nearrow"] & X \end{tikzcd}$$

(partially) lax pullback

Examples: • Let $\mathbb{D}^0 = *$, $\mathbb{D}^n = \Sigma_1 \mathbb{D}^{n-1}$.

These are the categorical disks, and boundary "spheres" $\partial \mathbb{D}^n = \Sigma_1 \partial \mathbb{D}^{n-1}$, $\partial \mathbb{D}^0 = \emptyset$.

Note that $\mathrm{id} \to \Omega\Sigma^1_1$ is an equiv,

counit is not: $\Sigma^1_1\Omega \to \mathrm{id}$

$$\Sigma^1_1 \underline{\mathrm{Map}}_X(0,1) \to X \quad \text{fully faithful}$$

- Also have the reduced spheres, $\mathbb{S}^n = \mathbb{D}^n/\partial\mathbb{D}^n$.

$$\mathbb{S}^1 = \{0 \to 1\}/\{0=1\} = *^{\circlearrowleft \mathbb{N}} = B\mathbb{N}$$

Similarly, $\mathbb{S}^n = \mathbb{B}^n\mathbb{F}^n$, $\mathbb{F}^n$ is the free $\mathbb{E}_n$-monoid on $*$. $\mathbb{F}^n$ is related to configurations of pts in $\mathbb{R}^n$.

$$\begin{tikzcd} \{0,1\} = \Sigma^1_1 \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & \Sigma^1_1 X \arrow[d] \\ * \arrow[r] & \tilde{\Sigma}^1_1 X \end{tikzcd}$$

$\tilde{\Sigma}^1_1 : \infty\mathrm{Cat} \to \infty\mathrm{Cat}_*$ has a right adjoint

$\tilde{\Omega} : \infty\mathrm{Cat}_* \to \infty\mathrm{Cat}$, $X \mapsto \underline{\mathrm{Map}}_X(*,*) = \underline{\mathrm{End}}_X(*)$.

unit $\mathrm{id} \to \tilde{\Omega}\tilde{\Sigma}^1_1$ is not an equiv.

$$\tilde{\Omega}\tilde{\Sigma}^1_1 = \mathbb{F}(-).$$

Similarly, $\tilde{\Sigma}^{1,n}_1 = \mathbb{B}^n\mathbb{F}^n(-)$

Then $\tilde{\Omega}^n\tilde{\Sigma}^{1,n}_1 X = \mathbb{F}^n(X) \xleftarrow{\not\simeq} X$.

Can invert that unit map, and obtain a higher version of spectra, where

$X = \{X_n\}$ $\qquad X_n \xrightarrow{\sim} \mathrm{End}_{X_{n+1}}(*)$.

Note: each X_n inherits the structure of a sym mon. ∞Cat.

Thm: (Masuda) There is a lax Gray smash product of categorical spectra, compatible w/ $\Sigma^{\infty}_{!,+} : (\infty\mathrm{Cat}, \boxtimes) \to (\infty\mathrm{Sp}, \wedge)$.
and this refines the $\otimes$-product of sym mon ∞-cats.

What is the Gray tensor?

$$\begin{array}{ccc} \infty\mathrm{Cat} \times \infty\mathrm{Cat} & \longrightarrow & \infty\mathrm{Cat} \\ (X, Y) & \longmapsto & X \boxtimes Y \\ & & \downarrow \\ & & X \times Y \end{array}$$

(unit is $*$)

E.g. $\mathbb{D}^1 \boxtimes \mathbb{D}^1 = \square^2$

$$= \left\{ \begin{array}{ccc} 00 & \to & 01 \\ \downarrow & \Rightarrow & \downarrow \\ 10 & \to & 11 \end{array} \right\}.$$

$$\begin{tikzcd} \partial\mathbb{D}^1 \boxtimes X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & \mathbb{D}^1 \boxtimes X \arrow[d] \\ \partial\mathbb{D}^1 \arrow[r] & \Sigma^1_1 X \end{tikzcd}$$

Gray tensor is $m\mathrm{Cat} \times n\mathrm{Cat} \xrightarrow{\boxtimes} (m+n)\mathrm{Cat}$

To understand Gray tensor, other lax operations, find (strong) generators of $\infty\mathrm{Cat}$.

<u>Def'n</u>: A small full subcat $\Theta \subset \infty\mathrm{Cat}$ is dense if $\infty\mathrm{Cat} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{P}(\Theta) = \mathrm{Fun}(\Theta^{op}, \mathcal{S})$ is fully faithful
($\mathcal{S} = \infty\mathrm{Gpd}$)

(automatic that a left adjoint exists)

Then this gives a presentation for $\infty\mathrm{Cat}$ by generators (objects of Θ) and relations (maps inverted by L).

<u>Rmk</u>: $1\mathrm{Cat} \underset{\text{space of objects}}{\overset{\text{inclusion}}{\rightleftarrows}} 0\mathrm{Cat} = \infty\mathrm{Gpd}$ ($\perp$)

This induces $(n+1)\mathrm{Cat} \rightleftarrows n\mathrm{Cat}$, and

$$\infty\mathrm{Cat} = \lim_{\text{in } 1\mathrm{CAT}} \{ \cdots \to n\mathrm{Cat} \to (n-1)\mathrm{Cat} \to \cdots \}$$

If $\mathcal{E}$ is some enriching category, $\mathfrak{D} \subset \mathcal{E}$ dense, $\mathcal{E} \overset{L}{\subset} \mathcal{P}(\mathfrak{D})$, and $\Delta \wr \mathcal{E} \subset \mathrm{Cat}(\mathcal{E})$. is dense.

Get thm of Rezk, $n\mathrm{Cat} \supset \Delta^{\wr n} = \Theta_n$.

And $\Theta = \operatorname{colim}_{n \to \infty} \Theta_n \subset \infty\mathrm{Cat}$ is dense.

What are some other dense subcats?

<u>Thm.</u> (Campion) $\Theta \subset \infty\mathrm{Cat}$ as the smallest full subcat containing $*$, closed under Σ, and $\vee$: $\infty\mathrm{Cat}_{\{0,1\}/} \ni X +_{\{1\}} Y \in \infty\mathrm{Cat}_{\{1,2\}/}$

One way to verify that $\mathfrak{D} \subset \infty\mathrm{Cat}$ is dense is to check $\Theta \subset \mathrm{Idem}(\mathfrak{D})$.

<u>Thm:</u> $\square = \{\square^n\}_{n \in \mathbb{N}} \underset{\text{constructed combinatorially}}{\subset} \infty\mathrm{Cat}^{\text{strict}} \subset \infty\mathrm{Cat}$ is dense because $\Theta \subset \mathrm{Idem}(\square)$.

Cause $\square^m \boxtimes \square^n = \square^{m+n}$,

$\infty\mathrm{Cat} \overset{L}{\leftrightarrows} \mathcal{P}(\square)$ creates the Gray tensor, monoidal for convolution on $\mathcal{P}(\square)$.

on $\infty\mathrm{Cat}$, via $X \boxtimes Y = L(RX \boxtimes RY)$.

Differently, $\square$ is still big,

$\Delta \subset \infty\mathrm{Cat}$ "oriented simplices" (aka "orientals") of R. Street, significantly smaller than $\square$,

$$\begin{tikzcd} \Delta^n \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & \square^n \arrow[r, two heads] & \Delta^n \end{tikzcd}.$$

<u>Thm</u>: (G.–Heine) $\Delta \subset \infty\mathrm{Cat}$ are dense in $\infty\mathrm{Cat}$.

Eg. $\Delta^2 \longrightarrow \Delta^2$ encodes the faces of the orientals.

$$\begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \longmapsto \begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur, bend left] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Uparrow"] & & 2 \end{tikzcd}$$

The orientals encode the lax join,

$$\begin{tikzcd} X \boxtimes \partial\Delta^1 \boxtimes Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & X \boxtimes \Delta^1 \boxtimes Y \arrow[d] \\ X + Y \arrow[r] & X \star Y. \end{tikzcd} \qquad (X * Y)$$

Check: $\Delta^m \star \Delta^n = \Delta^{m+n+1}$.

$$X \star - : \infty\mathrm{Cat} \longrightarrow \infty\mathrm{Cat}_{X/}$$
$$- \star Y : \infty\mathrm{Cat} \longrightarrow \infty\mathrm{Cat}_{Y/}.$$

These admit right adjoints,

$$Z \longmapsto Z_{X//}$$
$$Z \longmapsto Z_{//Y}.$$