Graded T-duality with H-flux for $2d$ σ-models

Fei Han

National University of Singapore

From Analysis to Homotopy Theory
A conference in honor of Uli's 60th birthday
May 13-17, 2024@Greifswald

joint with Varghese Mathai :

F. Han and V. Mathai, *T-duality with H-flux for 2d σ-models*, arXiv :2207.03134.

Developed from and motivated by the following papers as well as the papers listed in the Reference :

1. [A] M. F. Atiyah, *Circular symmetry and stationary-phase approximation*, Astérisque, 131 (1985), 43-59.

2. [B] J-M. Bismut, *Index theorem and equivariant cohomology on the loop space*, Comm. Math. Phys. 98 (1985), no. 2, 213-237.

3. [BEM] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality : Topology Change from H-flux*, Comm. Math. Phys. 249 (2004), 383-415.

Topological T-duality with H-flux

Consider such a pair :

$$\begin{array}{ccc} (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\ & {}_{\pi}\searrow \quad \swarrow_{\hat{\pi}} & \\ & X & \end{array}$$

Z and $\widehat{Z}$ are principal circle bundles over a manifold X;
A and $\widehat{A}$ are connections on them respectively ;
H is a background flux, i.e. a closed 3-form on Z with $\mathbb{Z}$ periods, and similarly for $\widehat{H}$

subject to the following relations :

$$\pi_!(H) = F^{\widehat{A}}, \quad \hat{\pi}_!(\widehat{H}) = F^{A},$$

F^A and $F^{\widehat{A}}$ are curvatures of A and $\widehat{A}$ respectively.

Topological T-duality with H-flux

$$\begin{array}{ccc} (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\ & {}_{\pi}\searrow \quad \swarrow_{\hat{\pi}} & \\ & X & \end{array}$$

$$\pi_!(H) = F^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = F^{A}$$

the model of topological T-duality with H-flux

More generally, there is the model of pair of principal torus bundles

Topological T-duality with H-flux

Related to the *Strominger-Yau-Zaslow Conjecture* concerning realization of Calabi–Yau manifolds and their mirrors as torus bundles over same base, proposed in their paper *Mirror Symmetry is T-duality*.

Foundational work on topological T-duality by Bunke, Schick, Nikolaus, Waldorf...

Topological T-duality with H-flux

For the T-dual pair with H-flux,

$$\begin{array}{ccc} (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\ & \searrow^{\pi} \quad {}^{\hat{\pi}}\swarrow & \\ & X & \end{array}$$

$$\pi_!(H) = F^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = F^{A}$$

several dual results, to just name a few, have been proved, verify or coincide with predictions of physicists :

Topological T-duality with H-flux

(1) Twisted cohomology

Let M be a smooth manifold, ω a closed 3 form on M. Consider the twisted de Rham complex

$$(\Omega^*(M), d+\omega)$$

and the cohomology of this complex $H^*(M,\omega)$. They are $\mathbb{Z}_2$-graded.

In the case of $M = Z$ and $\omega = H$,

$G \in \Omega^\bullet(Z)^{\mathbb{T}}$, the total RR fieldstrength,

$$\begin{array}{ccc} G \in \Omega^{even}(Z)^{\mathbb{T}} & \text{for} & \text{Type IIA}\,; \\ G \in \Omega^{odd}(Z)^{\mathbb{T}} & \text{for} & \text{Type IIB}. \end{array}$$

Topological T-duality with H-flux

To relate the Z and $\widehat{Z}$ sides, a fundamental construction is the correspondence space

$$
\begin{array}{ccccc}
 & & (Z \times_X \widehat{Z}, [p^* H] = [\widehat{p}^* \widehat{H}]) & & \\
 & \swarrow^{p} & & \searrow^{\widehat{p}} & \\
(Z, A, H) & & & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\
 & \searrow_{\pi} & & \swarrow_{\widehat{\pi}} & \\
 & & X & &
\end{array}
$$

Topological T-duality with H-flux

In [BEM], it shows that the Hori map

(1) $$T_H G = \int_{\mathbb{T}} e^{A\wedge \widehat{A}}\; G\,,$$

a Fermionic Fourier transformation through the correspondence space, gives

$$T_H\colon \Omega^{\bar{k}}(Z)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\widehat{Z})^{\widehat{\mathbb{T}}},$$

for $k = 0, 1$, (where $\bar{k}$ denotes the parity of k) is isomorphism, inducing isomorphism on twisted cohomology groups,

$$T_H : H^*(Z, H) \xrightarrow{\cong} H^{*+1}(\widehat{Z}, \widehat{H}).$$

Topological T-duality with H-flux

(2) Twisted K-theory

Let M be a smooth manifold, ω a closed 3 form on M with integral period. Then one has the twisted K-theory $K(M, \omega)$.

There were vast development of twisted K-theory by the work of Donovan-Karoubi, Atiyah-Segal, Freed-Hopkins-Teleman.

The twisted Chern classes for twisted K-theory have been studied by Atiyah-Segal (using Atiyah-Hirzebruch spectral sequence), Bouwknegt-Carey-Mathai-Murray-Stevenson (Chern-Weil theory).

Quantization of the twisted Chern class lead to the Mathai-Melrose-Singer *Fractional Index Theory* in the torsion case.

In particular, there is a twisted Chern character map :

$$\mathrm{Ch}_\omega : K^*(M, \omega) \to H^*(M, \omega).$$

D-brane charges in Type IIA String theory are classified by twisted K-theory $K^0(Z, H)$ and in Type IIB String theory are classified by twisted K-theory $K^1(Z, H)$ (Bouwknegt-Mathai 2000, Bouwknegt-Carey-Mathai-Murray-Stevenson 2002)

In [BEM], using the correspondence space, it shows that there is an isomorphism

$$T_K \colon K^*(Z, H) \to K^{*+1}(\widehat{Z}, \widehat{H}),$$

and moreover, there is commutative diagram,

$$\begin{array}{ccc} K^*(Z,H) & \xrightarrow{T_K} & K^{*+1}(\hat{Z},\hat{H}) \\ \Big\downarrow{\scriptstyle Ch_H} & & \Big\downarrow{\scriptstyle Ch_{\hat{H}}} \\ H^*(Z,H) & \xrightarrow[T_H]{} & H^{*+1}(\hat{Z},\hat{H}) \end{array} \tag{2}$$

Topological T-duality with H-flux

(3) Loop space perspective

Atiyah-Witten-Bismut's work ([A], [B]) studied equivariant cohomolgy of free loop spaces, and formally realized the Atiyah-Singer index theory as fixed point theory on free loop spaces.

Indicates that T-duality with H-flux and the Hori formulae for spacetime should be a shadow of T-duality and Hori formulae for loop space of spacetime.

Along this free loop space perspective, we mention some work :

A. Linshaw and V. Mathai, *Twisted Chiral De Rham Complex, Generalized Geometry, and T-duality*, Comm. Math. Phys., 339, No. 2, (2015).

F. Han and V. Mathai, *Exotic twisted equivariant cohomology of loop spaces, twisted Bismut-Chern character and T-duality*, Comm. Math. Phys., 337, no. 1, (2015) 127–150.

Topological T-duality with H-flux

We highlight that, mathematically we obtained a delocalization of the twisted cohomology $H^*(Z, H)$, from Z, the S^1-fixed point set of LZ, to LZ,

$$Z \rightsquigarrow LZ$$

$$H^*(Z, H) \xrightarrow{\cong} H^*(LZ, \mathcal{L}_H),$$

where $\mathcal{L}_H$ is the holonomy line bundle on the loop space LZ arising from the flux or gerbe H on Z.

Natural Questions : (1) what's the data on the double loop space LLZ arising from the flux or gerbe H on Z?
(2) is there a delocalization from Z to the double loop space LLZ?

$$Z \rightsquigarrow LLZ?$$

Topological T-duality with H-flux

(4) Double loop spaces perspective

The main topic for this talk.

Double loop the T-dual pair, we have the following picture :

$$\begin{array}{ccc}
(\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}}) & & (\mathcal{L}_{\widehat{H},\tau}, \nabla^{\mathcal{L}_{\widehat{H},\tau}}) \\
\downarrow & & \downarrow \\
C^\infty(T^2, Z)^H & & C^\infty(T^2, \widehat{Z})^{\widehat{H}} \\
& \searrow^{LL\pi} \quad {}^{LL\widehat{\pi}}\swarrow & \\
& C^\infty(T^2, X) &
\end{array}$$

where T^2 is the 2-dimensional torus, $C^\infty(T^2, Z) = LLZ$ is the double loop space, $C^\infty(T^2, Z)^H$ is certain circle bundle over the double loop space, τ is a modular parameter, $(\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}})$ is the average τ-holonomy line bundle with a canonical connection. Similarly notations on the dual side.

Topological T-duality with H-flux

We will explain the notations in more details later, construct some complexes from the objects $(\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}}) \to C^\infty(T^2, Z)^H$ and establish the T-dual map between the Z side and $\widehat{Z}$ side.

This result in some sense is the T-duality with H-flux for $2d$ σ-model, answering a question of Hori.

Double loop spaces

To study T-duality from the perspective of 2d, joint with Mathai, we give relevant constructions and discover some properties on double loop spaces.

A. we introduce the double loop Bryinski cover for M :
$\{\mathsf{U}_\alpha\}$: maximal open cover of M with the property that $H^i(U_{\alpha_I}) = 0$ for $i \geq 3$, where $U_{\alpha_I} = \bigcap_{i\in I} U_{\alpha_i},\ |I| < \infty$.

In fact, let $x : T^2 \to M$ be a smooth loop in M and U_x a tubular neighbourhood of x in M. $\{LL\mathsf{U}_x, x \in LLM\}$ covers LLM.

Double loop spaces

B. we construct various transgression maps or averaging maps :

Let ev is the evaluation map

$$ev : LLM \times T^2 \to M : (x, s, t) \mapsto x(s, t),$$

we have the double transgression map :

$$\mu_{1,2} : \Omega^{\bullet}(\mathsf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-2}(LL\mathsf{U}_{\alpha_I})$$

defined by

$$\mu_{1,2}(\xi_I) = \int_{T^2} ev^*(\xi_I), \qquad \xi_I \in \Omega^{\bullet}(\mathsf{U}_{\alpha_I}).$$

Double loop spaces

we have the averaging after transgression map :

$$\overline{\mu_1}^2 : \Omega^\bullet(\mathsf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LL\mathsf{U}_{\alpha_I})$$

defined by

$$\overline{\mu_1}^2(\xi_I) = \int_{S^1}\left(\int_{S^1} ev^*(\xi_I)\right)dt, \qquad \xi_I \in \Omega^\bullet(\mathsf{U}_{\alpha_I}),$$

i.e. integrate $ev^*(\xi_I)$ along the first circle and then average along the second circle. Similarly, one has

$$\overline{\mu_2}^1 : \Omega^\bullet(\mathsf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LL\mathsf{U}_{\alpha_I}).$$

Double loop spaces

Let $\omega \in \Omega^i(M)$. One also has the double loop averaging map

$$\overline{\overline{\omega}} := \int_{T^2} ev^*(\omega) ds \wedge dt \in \Omega^i(LLM).$$

Clearly $L_{K_i}\overline{\overline{\omega}} = 0, i = 1, 2$. Moreover it is not hard to see that

$$d\overline{\overline{\omega}} = \overline{\overline{d\omega}}, \quad \mu_{1,2}(\omega) = \iota_{K_2}\iota_{K_1}\overline{\overline{\omega}}.$$

In addition to evaluation map (3), there are also partial evaluation maps

$$ev_1 : LLM \times S^1 \to LM, \quad (x, s) \mapsto x(s, *),$$

$$ev_2 : LLM \times S^1 \to LM, \quad (x, t) \mapsto x(*, t),$$

and certain projections from double loops to one loops

$$\pi_i : LLM \to LM, \; i = 1, 2$$

defined by $\pi_1 = ev_2|_{t=0}$, i.e restriction to the first circle and $\pi_2 = ev_1|_{s=0}$, i.e restriction to the second circle.

Double loop spaces

C. we construct the average holonomy line bundle on the double loop space arising from the following data.

Suppose M carries a gerbe with connection $(H, B_\alpha, F_{\alpha\beta}, (L_{\alpha\beta}, \nabla^{L_{\alpha\beta}}))$, with $H \in \Omega^3(M)$, $B_\alpha \in \Omega^2(\mathsf{U}_\alpha)$ and $(L_{\alpha\beta}, \nabla^{L_{\alpha\beta}})$ being a complex line bundle over $\mathsf{U}_{\alpha\beta} = \mathsf{U}_\alpha \cap \mathsf{U}_\beta$ such that

$$H = dB_\alpha \text{ on } \mathsf{U}_\alpha,$$

$$B_\beta - B_\alpha = F_{\alpha\beta} = \left(\nabla^{L_{\alpha\beta}}\right)^2 \quad \text{on } \mathsf{U}_\alpha \cap \mathsf{U}_\beta,$$

$$(L_{\alpha\beta}, \nabla^{L_{\alpha\beta}}) \otimes (L_{\beta\gamma}, \nabla^{L_{\beta\gamma}}) \otimes (L_{\gamma\alpha}, \nabla^{L_{\gamma\alpha}}) \simeq (\mathbb{C}, d) \text{ on } \mathsf{U}_\alpha \cap \mathsf{U}_\beta \cap \mathsf{U}_\gamma.$$

Double loop spaces

Let $\mathcal{L}$ be the holonomy line bundle on LM arising from H.

Let $\tau \in \mathbb{H}$, the upper half plane. This means now we consider the complex structures on the source $T^2 \to M$.

Roughly speaking, our construction of the τ-average holonomy line bundle is to make sense of the following line bundle over LLM

$$\pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L})^{\otimes \tau},$$

i.e. $\pi_1^*(\mathcal{L})\otimes$ tensored with the "τ-th power" of $\pi_2^*(\mathcal{L})^{\otimes \tau}$, similar to $a + b\tau \in \mathbb{C}$.

Unfortunately, since τ is not an integer, but a complex number with positive imaginary part, $\pi_2^*(\mathcal{L})^{\otimes \tau}$ does not make sense.

Double loop spaces

In the following, we will explain a situation that can make sense out of this. Let ξ be a complex line bundle over a manifold X.

Let $\mathfrak{U} = \{U_\alpha\}$ be an open good cover of X. Let $\{g_{\alpha\beta}\}$ be a system of $U(1)$-valued transition functions w.r.t $\mathfrak{U}$. This gives us a closed Cech cocycle $\{\theta_{\alpha\beta}\}$ valued in $\mathbb{R}/\mathbb{Z}$ by taking $\theta_{\alpha\beta} = \frac{1}{2\pi i}\ln g_{\alpha\beta}$ in the argument interval $[0, 2\pi)$. So $\{\theta_{\alpha\beta}\} \in C^1(\mathfrak{U}, \mathbb{R}/\mathbb{Z})$.

Double loop spaces

Let

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

be the obvious exact sequence. It is not hard to show that

(1) $\{\theta_{\alpha\beta}\}$ can lifted as the image of a Cech cocyle in $C^1(\mathfrak{U}, \mathbb{R}) \iff \xi$ is trivial.
(2) Different liftings differ by δc for some $c \in C^1(\mathfrak{U}, \mathbb{Z})$.

Let ξ be trivial and $\eta_{\alpha\beta} \in C^1(\mathfrak{U}, \mathbb{R})$ be a lifting of $\{\theta_{\alpha\beta}\}$. Then we can consider the $\mathbb{C}$-valued functions $\{e^{2\pi i\tau\eta_{\alpha\beta}}\}$, which satisfying

$$e^{2\pi i\tau\eta_{\alpha\beta}} \cdot e^{2\pi i\tau\eta_{\beta\gamma}} \cdot e^{2\pi i\tau\eta_{\gamma\alpha}} = 1$$

and therefore glues us a complex line bundle over X. One can consider it as the τ-th power of ξ. One may think of this construction by taking $\tau = \frac{1}{n}$, $n \in \mathbb{Z}$ and the construction of an n-th root of ξ.

Double loop spaces

Let us come back to the double loop space. Although $\pi_2^*(\mathcal{L})^{\otimes\tau}$ does not make sense, but suppose

$$p : \mathcal{S} \to LLM$$

be the circle bundle of $\pi_2^*(\mathcal{L})$, the pull back $p^*(\pi_2^*(\mathcal{L}))$ is a trivial bundle over $\mathcal{S}$. Then $p^*(\pi_2^*(\mathcal{L}))^{\otimes\tau}$ makes sense as explained above. So on $\mathcal{S}$, we have the line bundle

$$p^*(\pi_1^*(\mathcal{L})) \otimes p^*(\pi_2^*(\mathcal{L}))^{\otimes\tau}.$$

This is just the τ-average holonomy line bundle we are going to construct.

Double loop spaces

Note that on LLM, there is the T^2-action. Hence we have to give a T^2-invariant transition function for the line bundle

$$p^*(\pi_1^*(\mathcal{L})) \otimes p^*(\pi_2^*(\mathcal{L}))^{\otimes \tau}.$$

To achieve this, we have to give $(K_1 + \tau K_2)$-invariant transition functions for this line bundle, which we will explain in the following.

Double loop spaces

For any double loop $x \in LL\mathsf{U}_\alpha \cap LL\mathsf{U}_\beta$, i.e. $x : T^2 \to \mathsf{U}_{\alpha\beta}$, denote the holonomy of the $\nabla^{L_{\alpha\beta}}$ along the K_1-direction of by hol^1, which is a function of t; and the holonomy of the $\nabla^{L_{\alpha\beta}}$ along the K_2-direction by hol^2, which is a function of s.

Consider the function on $LL\mathsf{U}_\alpha \cap LL\mathsf{U}_\beta$

$$g_{\alpha\beta} := e^{\overline{\ln hol^1_{\alpha\beta}}^2} \cdot e^{\tau \overline{\ln hol^2}^1_{\alpha\beta}}.$$

Note here for $\ln hol^1$, it is continuously defined for $t \in [0, 1)$, and for $\ln hol^2$, it is continuously defined for $s \in [0, 1)$.

Double loop spaces

$$
\begin{aligned}
& L_{K_1+\tau K_2} g_{\alpha\beta} \\
=& L_{K_1+\tau K_2}\left(e^{\overline{\ln hol^1_{\alpha\beta}}^2}\cdot e^{\tau\overline{\ln hol^2}^1_{\alpha\beta}}\right) \\
=& e^{\overline{\ln hol^1_{\alpha\beta}}^2}\cdot e^{\tau\overline{\ln hol^2}^1_{\alpha\beta}}[L_{\tau K_2}\overline{\ln hol^1_{\alpha\beta}}^2 + L_{K_1}\tau\overline{\ln hol^2}^1_{\alpha\beta}] \\
=& e^{\overline{\ln hol^1_{\alpha\beta}}^2}\cdot e^{\tau\overline{\ln hol^2}^1_{\alpha\beta}}[\tau\overline{K_2\ln hol^1_{\alpha\beta}}^2 + \tau\overline{K_1\ln hol^2}^1_{\alpha\beta}] \\
=& e^{\overline{\ln hol^1_{\alpha\beta}}^2}\cdot e^{\tau\overline{\ln hol^2}^1_{\alpha\beta}}\tau[\overline{\iota_{K_2}d\ln hol^1_{\alpha\beta}}^2 + \overline{\iota_{K_1}d\ln hol^2}^1_{\alpha\beta}] \\
=& e^{\overline{\ln hol^1_{\alpha\beta}}^2}\cdot e^{\tau\overline{\ln hol^2}^1_{\alpha\beta}}\tau\left[\overline{\iota_{K_2}\iota_{K_1}\overline{F_{\alpha\beta}}^1}^2 + \overline{\iota_{K_1}\iota_{K_2}\overline{F_{\alpha\beta}}^2}^1\right] \\
=& e^{\overline{\ln hol^1_{\alpha\beta}}^2}\cdot e^{\tau\overline{\ln hol^2}^1_{\alpha\beta}}\tau\left[\iota_{K_1}\iota_{K_2}\overline{\overline{F_{\alpha\beta}}} + \iota_{K_2}\iota_{K_1}\overline{\overline{F_{\alpha\beta}}}\right] \\
=& 0.
\end{aligned}
$$

So $g_{\alpha\beta}$ is $(K_1+\tau K_2)$-invariant.

Double loop spaces

Denote

$$h_{\alpha\beta\gamma} := g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha}.$$

On the triple intersection $LL\mathsf{U}_\alpha \cap LL\mathsf{U}_\beta \cap LL\mathsf{U}_\gamma$,

$$hol^i_{\alpha\beta} hol^i_{\beta\gamma} hol^i_{\gamma\alpha} = 1, \ \ i = 1, 2.$$

Hence $\ln hol^i_{\alpha\beta} + \ln hol^i_{\beta\gamma} + \ln hol^i_{\gamma\alpha} \in 2\pi i\mathbb{Z}$. As $\ln hol$'s are continuously defined, one must have

$$h_{\alpha\beta\gamma} = e^{2\pi i m_{\alpha\beta\gamma}\tau} \tag{3}$$

for some $m_{\alpha\beta\gamma} \in \mathbb{Z}$, where $\{m_{\alpha\beta\gamma}\}$ forms the Cech cocycle representing $\pi_2^*(c_1(\mathcal{L}_B))$ in $H^2(LLM, \mathbb{Z})$ with $c_1(\mathcal{L}_B)$ being the first Chern class of the holonomy line bundle $\mathcal{L}_B$ on LM arising from the ω on M.

Double loop spaces

Let $p : \mathcal{S}_B \to LM$ be the circle bundle of the line bundle $\mathcal{L}_B \to LM$. Then $p^*\mathcal{L}_B$ is a trivial line bundle over $\mathcal{S}_B$. Hence the class $p^*(c_1(\mathcal{L}_B))$ be 0 on $\mathcal{S}_B$. Therefore $\widetilde{\pi}_2^* \circ p^*(c_1(\mathcal{L}_B))$ is 0 on $p^*\mathcal{S}_B$, the pulled back circle bundle over LLM.

$$\begin{array}{ccc} p^*\mathcal{S}_B & \xrightarrow{\widetilde{\pi}_2} & \mathcal{S}_B \\ \downarrow{\scriptstyle \widetilde{p}} & & \downarrow{\scriptstyle p} \\ LLM & \xrightarrow[\pi_2]{} & LM \end{array} \tag{4}$$

For simplicity, in the sequel we will denote the total space $p^*\mathcal{S}_B$ by LLM^ω, which carries the induced T^2-action arising from LLM.

Double loop spaces

Using $\{g_{\alpha\beta}\}$, after some treatment on LLM^ω, we can construct a complex line bundle $\mathcal{L}_{B,\tau}$ on LLM^ω, which we call average τ-holonomy line bundle, which also carries a canonical connection $\nabla^{\mathcal{L}_{B,\tau}}$ out from the B-field.

Double loop spaces

D. Equivariantly super flatness

Denote by $\Omega^\bullet_{bas}(LLM^\omega, \mathcal{L}_{H,\tau})$ the space of basic differential forms on LLM^ω with values in the average τ-holonomy line bundle $\mathcal{L}_{B,\tau}$. Here basic form means that contracted with vertical tangent vectors gives 0. Let u be an indeterminate such that $\deg u = 2$. Consider the odd operator

$$(5) \qquad Q_{B,\tau} := \nabla^{\mathcal{L}_{B,\tau}} - u\iota_{K_1+\tau K_2} + u^{-1}\widetilde{p}^*\overline{\overline{\omega}}$$

which acts on $\Omega^\bullet(LLM^\omega, \mathcal{L}_{B,\tau})_{bas}[[u, u^{-1}]]$.

Double loop spaces

Theorem (H.-Mathai)

The following identities hold,

$$\frac{1}{2}[Q_{B,\tau}, Q_{B,\tau}] = Q_{B,\tau}^2 = -uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}, \quad [Q_{B,\tau}, -uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}] = 0,$$

where $L_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}$ is the Lie derivative along the direction $K_1 + \tau K_2$.

So the odd operator $Q = Q_{B,\tau}$ and the even operator $P = -uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}$ obey the relations

$$\frac{1}{2}[Q, Q] = P, \quad [Q, P] = 0$$

of the superalgebra considered in Witten's paper *Supersymmetry and Morse theory*.

Double loop spaces

E. Localization

The above theorem tells us there is a complex

$$\left(\Omega^{\bullet}_{bas}(LLM^{\omega}, \mathcal{L}_{B,\tau})^{K_1+\tau K_2}[[u, u^{-1}]], Q_{B,\tau}\right).$$

Note that the zeros of the complex vector field $K_1 + \tau K_2$ are T^2-fixed points of LLM, i.e.M. We have the Borel-Witten type localization :

Theorem (H.-Mathai)

Let $i : M \times S^1 \to LLM^{\omega}$ be the inclusion map. Then the restriction map

$$i^* : \left(\Omega^{\bullet}_{bas}(LLM^{\omega}, \mathcal{L}_{B,\tau})^{K_1+\tau K_2}[[u, u^{-1}]], Q_{B,\tau}\right)$$
$$\to \left(\Omega^{\bullet}_{bas}(M \times S^1)[[u, u^{-1}]], d + u^{-1}p^*\omega\right) \cong \left(\Omega^{\bullet}(M)[[u, u^{-1}]], d + u^{-1}\omega\right)$$

is a quasi-isomorphism, $\forall \tau \in \mathbb{H}$.

Now we apply our theory on double loop spaces to study T-duality from the perspective of 2d

Recall that we have the T-dual pair with H-flux :

$$\begin{array}{ccc} (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\ & \searrow^{\pi} \quad {}^{\hat{\pi}}\swarrow & \\ & X & \end{array}$$

$$\pi_!(H) = F^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = F^{A}$$

Double loop the T-dual pair, we have the following picture :

$$\begin{array}{ccc} (\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}}) & & (\mathcal{L}_{\widehat{H},\tau}, \nabla^{\mathcal{L}_{\widehat{H},\tau}}) \\ \downarrow & & \downarrow \\ C^\infty(T^2, Z)^H & & C^\infty(T^2, \widehat{Z})^{\widehat{H}} \\ & \searrow^{LL\pi} \quad {}^{LL\widehat{\pi}}\swarrow & \\ & C^\infty(T^2, X) & \end{array}$$

A sheaf over upper half plane $\mathbb{H}$

For the pair (Z, H), define a sheaf $(\mathrm{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H), \mathcal{Q}_H)$ on $\mathbb{H}$ of commutative differential graded algebras that to $U \subset \mathbb{H}$ assigns the graded complex of $\mathcal{O}(U)$-modules

$$(\mathrm{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H)(U), \mathcal{Q}_H)$$
$$:= \bigoplus_{m\in\mathbb{Z}} \left(\mathcal{O}(U; \Omega_{bas}^{\bullet,\mathbb{T}}(C^\infty(T^2, Z)^H, \mathcal{L}_{H,\tau}^{\otimes m})[[u, u^{-1}]]^{K_1+\tau K_2}) \cdot y^m,\ \ Q_{mB,\tau}\right),$$

where $\mathcal{O}(U; \Omega_{bas}^{\bullet,\mathbb{T}}(C^\infty(T^2, Z)^H, \mathcal{L}_{H,\tau}^{\otimes m})[[u, u^{-1}]]^{K_1+\tau K_2})$ means i.e for each $\tau \in U$, one assigns to it an element in $\Omega_{bas}^{\bullet,\mathbb{T}}(C^\infty(T^2, Z)^H, \mathcal{L}_{H,\tau}^{\otimes m})[[u, u^{-1}]]^{K_1+\tau K_2}$. Dually, one can also define the sheaf $(\mathrm{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}), \mathcal{Q}_{\widehat{H}})$.

Note that we have assembled mH for $m \in \mathbb{Z}$, i.e. we consider the graded version. Here y is a formal variable to keep track of the level m by y^m.

Passing to cohomology, we get the sheaves $\mathcal{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H)$ and $\mathcal{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}})$. The localisation theorem tells us that the restriction maps

$$res : \mathcal{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H) \to \mathcal{G}(Z, H), \quad \widehat{res} : \mathcal{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}) \to \mathcal{G}(\widehat{Z}, \widehat{H})$$

are isomorphisms of sheaves.

In a previous work joint with Mathai, we constructed the **graded Hori morphisms** between the sheaves

$$GHor_* : (\mathrm{G}(Z, H), D^H) \to (\mathrm{G}(\widehat{Z}, \widehat{H}), D^{\widehat{H}}), \;\; GHor : \mathcal{G}(Z, H) \to \mathcal{G}(\widehat{Z}, \widehat{H}).$$

$$\widehat{GHor}_* : (\mathrm{G}(\widehat{Z}, \widehat{H}), D^{\widehat{H}}) \to (\mathrm{G}(Z, H), D^H), \;\; \widehat{GHor} : \mathcal{G}(\widehat{Z}, \widehat{H}) \to \mathcal{G}(Z, H).$$

and showed that they send Jacobi forms to Jacobi forms.

Now we construct the **graded Hori morphisms for $2d$ σ-models** by

$$GHor^{\sigma} := \widehat{res}^{-1} \circ GHor \circ res : \mathcal{G}(C^{\infty}(T^2, Z)^H, \mathcal{L}_H) \to \mathcal{G}(C^{\infty}(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}),$$
$$\widehat{GHor}^{\sigma} := res^{-1} \circ \widehat{GHor} \circ \widehat{res} : \mathcal{G}(C^{\infty}(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}) \to \mathcal{G}(C^{\infty}(T^2, Z)^H, \mathcal{L}_H),$$

assembled in the following commutative diagram,

$$\begin{array}{ccc}
\mathcal{G}(C^{\infty}(T^2, Z)^H, \mathcal{L}_H) & \underset{\widehat{GHor}^{\sigma}}{\overset{GHor^{\sigma}}{\rightleftarrows}} & \mathcal{G}(C^{\infty}(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}) \\
\downarrow {\scriptstyle res\, \cong} & & \downarrow {\scriptstyle \widehat{res}\, \cong} \\
\mathcal{G}(Z, H) & \underset{\widehat{GHor}}{\overset{GHor}{\rightleftarrows}} & \mathcal{G}(\widehat{Z}, \widehat{H})
\end{array}$$

Theorem (H.-Mathai)

One has

$$(6) \qquad \widehat{GHor}^{\sigma} \circ GHor^{\sigma} = -y\frac{\partial}{\partial y}, \quad GHor^{\sigma} \circ \widehat{GHor}^{\sigma} = -y\frac{\partial}{\partial y}.$$

References

1. E. Witten, *Supersymmetry and Morse theory*. J. Differential Geom., **17** no.4(1983) 661-692.

2. A. Strominger, S.T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Physics B Volume 479, Issues 1–2, 11 November 1996, Pages 243-259.

3. K. Hori and C. Vafa, *Mirror symmetry*, hep-th/0002222.

4. F. Han and V. Mathai, *Exotic twisted equivariant cohomology of loop spaces, twisted Bismut-Chern character and T-duality*, Comm. Math. Phys., 337, no. 1, (2015) 127–150.

5. Linshaw and V. Mathai, *Twisted Chiral De Rham Complex, Generalized Geometry, and T-duality*, Comm. Math. Phys., 339, No. 2, (2015).

6. F. Han and V. Mathai, *T-Duality, Jacobi Forms and Witten Gerbe Modules*, Adv. Theo. Math. Phys., 25 no. 5 (2021) 1235-1266.

Happy Birthday, Uli !