Graded T-duality with H-flux for $2d \sigma$ -models

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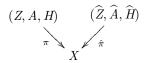
From Analysis to Homotopy Theory A conference in honor of Uli's 60th birthday May 13-17, 2024@Greifswald joint with Varghese Mathai:

F. Han and V. Mathai, T-duality with H-flux for 2d σ -models, arXiv :2207.03134.

Developed from and motivated by the following papers as well as the papers listed in the Reference :

- 1. [A] M. F. Atiyah, Circular symmetry and stationary-phase approximation, Astérisque, 131 (1985), 43-59.
- 2. [B] J-M. Bismut, Index theorem and equivariant cohomology on the loop space, Comm. Math. Phys. 98 (1985), no. 2, 213-237.
- 3. [BEM] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality : Topology Change from H-flux*, Comm. Math. Phys. 249 (2004), 383-415.

Consider such a pair :



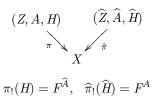
Z and \widehat{Z} are principal circle bundles over a manifold X; A and \widehat{A} are connections on them respectively:

H is a background flux, i.e. a closed 3-form on Z with $\mathbb Z$ periods, and similarly for $\widehat H$

subject to the following relations:

$$\pi_!(H) = F^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = F^A,$$

 F^A and $F^{\widehat{A}}$ are curvatures of A and \widehat{A} respectively.



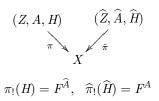
the model of topological T-duality with H-flux

More generally, there is the model of pair of principal torus bundles

Related to the *Strominger-Yau-Zaslow Conjecture* concerning realization of Calabi–Yau manifolds and their mirrors as torus bundles over same base, proposed in their paper *Mirror Symmetry is T-duality*.

Foundational work on topological T-duality by Bunke, Schick, Nikolaus, Waldorf...

For the T-dual pair with H-flux,



several dual results, to just name a few, have been proved, verify or coincide with predictions of physicists :

(1) Twisted cohomology

Let M be a smooth manifold, ω a closed 3 form on M. Consider the twisted de Rham complex

$$(\Omega^*(M), d+\omega)$$

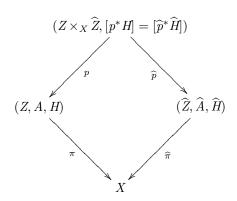
and the cohomology of this complex $H^*(M,\omega)$. They are \mathbb{Z}_2 -graded.

In the case of M = Z and $\omega = H$,

 $G \in \Omega^{\bullet}(Z)^{\mathbb{T}}$, the total RR fieldstrength,

$$G \in \Omega^{even}(Z)^{\mathbb{T}}$$
 for Type IIA;
 $G \in \Omega^{odd}(Z)^{\mathbb{T}}$ for Type IIB.

To relate the Z and \widehat{Z} sides, a fundamental construction is the correspondence space



In [BEM], it shows that the Hori map

(1)
$$T_H G = \int_{\mathbb{T}} e^{A \wedge \widehat{A}} G,$$

a Fermionic Fourier transformation through the correspondence space, gives

$$T_H: \Omega^{\overline{k}}(Z)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\widehat{Z})^{\widehat{\mathbb{T}}},$$

for k=0,1, (where \bar{k} denotes the parity of k) is isomorphism, inducing isomorphism on twisted cohomology groups,

$$T_H: H^*(Z, H) \xrightarrow{\cong} H^{*+1}(\widehat{Z}, \widehat{H}).$$

(2) Twisted K-theory

Let M be a smooth manifold, ω a closed 3 form on M with integral period. Then one has the twisted K-theory $K(M, \omega)$.

There were vast development of twisted K-theory by the work of Donovan-Karoubi, Atiyah-Segal, Freed-Hopkins-Teleman.

The twisted Chern classes for twisted K-theory have been studied by Atiyah-Segal (using Atiyah-Hirzebruch spectral sequence), Bouwknegt-Carey-Mathai-Murray-Stevenson (Chern-Weil theory).

Quantization of the twisted Chern class lead to the Mathai-Melrose-Singer Fractional Index Theory in the torsion case.

In particular, there is a twisted Chern character map:

$$\mathrm{Ch}_{\omega}: K^*(M,\omega) \to H^*(M,\omega).$$

D-brane charges in Type IIA String theory are classified by twisted K-theory $K^0(Z, H)$ and in Type IIB String theory are classified by twisted K-theory $K^1(Z, H)$ (Bouwknegt-Mathai 2000, Bouwknegt-Carey-Mathai-Murray-Stevenson 2002)

In [BEM], using the correspondence space, it shows that there is an isomorphism

$$T_K \colon K^*(Z, H) \to K^{*+1}(\widehat{Z}, \widehat{H}),$$

and moreover, there is commutative diagram,

(2)
$$K^{*}(Z, H) \xrightarrow{T_{K}} K^{*+1}(\hat{Z}, \hat{H})$$

$$\downarrow Ch_{\hat{H}}$$

(3) Loop space perspective

Atiyah-Witten-Bismut's work ([A], [B]) studied equivariant cohomolgy of free loop spaces, and formally realized the Atiyah-Singer index theory as fixed point theory on free loop spaces.

Indicates that T-duality with H-flux and the Hori formulae for spacetime should be a shadow of T-duality and Hori formulae for loop space of spacetime.

Along this free loop space perspective, we mention some work:

- A. Linshaw and V. Mathai, Twisted Chiral De Rham Complex, Generalized Geometry, and T-duality, Comm. Math. Phys., 339, No. 2, (2015).
- F. Han and V. Mathai, Exotic twisted equivariant cohomology of loop spaces, twisted Bismut-Chern character and T-duality, Comm. Math. Phys., 337, no. 1, (2015) 127–150.

We highlight that, mathematically we obtained a delocalization of the twisted cohomology $H^*(Z, H)$, from Z, the S^1 -fixed point set of LZ, to LZ,

$$Z \sim LZ$$

$$H^*(Z, H) \xrightarrow{\cong} H^*(LZ, \mathcal{L}_H),$$

where \mathcal{L}_H is the holonomy line bundle on the loop space LZ arising from the flux or gerbe H on Z.

Natural Questions: (1) what's the data on the double loop space LLZ arising from the flux or gerbe H on Z?

(2) is there a delocalization from Z to the double loop space LLZ?

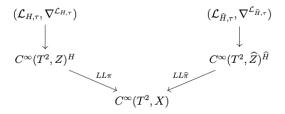
$$Z \sim LLZ$$
?



(4) Double loop spaces perspective

The main topic for this talk.

Double loop the T-dual pair, we have the following picture:



where T^2 is the 2-dimensional torus, $C^{\infty}(T^2, Z) = LLZ$ is the double loop space, $C^{\infty}(T^2, Z)^H$ is certain circle bundle over the double loop space, τ is a modular parameter, $(\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}})$ is the average τ -holonomy line bundle with a canonical connection. Similarly notations on the dual side.

We will explain the notations in more details later, construct some complexes from the objects $(\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}}) \to C^{\infty}(T^2, Z)^H$ and establish the T-dual map between the Z side and \widehat{Z} side.

This result in some sense is the T-duality with H-flux for 2d σ -model, answering a question of Hori.

To study T-duality from the perspective of 2d, joint with Mathai, we give relevant constructions and discover some properties on double loop spaces.

A. we introduce the double loop Bryinski cover for M: $\{U_{\alpha}\}$: maximal open cover of M with the property that $H^{i}(U_{\alpha_{I}}) = 0$ for $i \geq 3$, where $U_{\alpha_{I}} = \bigcap_{i \in I} U_{\alpha_{i}}, |I| < \infty$.

In fact, let $x: T^2 \to M$ be a smooth loop in M and U_x a tubular neighbourhood of x in M. $\{LL\mathsf{U}_x, x \in LLM\}$ covers LLM.

B. we construct various transgression maps or averaging maps:

Let ev is the evaluation map

$$ev: LLM \times T^2 \to M: (x, s, t) \mapsto x(s, t),$$

we have the double transgression map:

$$\mu_{1,2}: \Omega^{\bullet}(\mathsf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-2}(LL\mathsf{U}_{\alpha_I})$$

defined by

$$\mu_{1,2}(\xi_I) = \int_{T^2} ev^*(\xi_I), \qquad \xi_I \in \Omega^{\bullet}(\mathsf{U}_{\alpha_I}).$$

we have the averaging after transgression map:

$$\overline{\mu_1}^2: \Omega^{\bullet}(\mathsf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LL\mathsf{U}_{\alpha_I})$$

defined by

$$\overline{\mu_1}^2(\xi_I) = \int_{S^1} \left(\int_{S^1} ev^*(\xi_I) \right) dt, \qquad \xi_I \in \Omega^{\bullet}(\mathsf{U}_{\alpha_I}),$$

i.e. integrate $ev^*(\xi_I)$ along the first circle and then average along the second circle. Similarly, one has

$$\overline{\mu_2}^1: \Omega^{\bullet}(\mathsf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LL\mathsf{U}_{\alpha_I}).$$

Let $\omega \in \Omega^i(M)$. One also has the double loop averaging map

$$\overline{\overline{\omega}} := \int_{T^2} ev^*(\omega) ds \wedge dt \in \Omega^i(LLM).$$

Clearly $L_{K_i}\overline{\overline{\omega}}=0, i=1,2.$ Moreover it is not hard to see that

$$d\overline{\overline{\omega}} = \overline{\overline{d\omega}}, \quad \mu_{1,2}(\omega) = \iota_{K_2} \iota_{K_1} \overline{\overline{\omega}}.$$

In addition to evaluation map (3), there are also partial evaluation maps

$$ev_1: LLM \times S^1 \to LM, (x, s) \mapsto x(s, *),$$

$$ev_2: LLM \times S^1 \to LM, (x, t) \mapsto x(*, t),$$

and certain projections from double loops to one loops

$$\pi_i: LLM \to LM, i=1,2$$

defined by $\pi_1 = ev_2|_{t=0}$, i.e restriction to the first circle and $\pi_2 = ev_1|_{s=0}$, i.e restriction to the second circle.

C. we construct the average holonomy line bundle on the double loop space arising from the following data.

Suppose M carries a gerbe with connection $(H, B_{\alpha}, F_{\alpha\beta}, (L_{\alpha\beta}, \nabla^{L_{\alpha\beta}}))$, with $H \in \Omega^3(M), B_{\alpha} \in \Omega^2(\mathsf{U}_{\alpha})$ and $(L_{\alpha\beta}, \nabla^{L_{\alpha\beta}})$ being a complex line bundle over $\mathsf{U}_{\alpha\beta} = \mathsf{U}_{\alpha} \cap \mathsf{U}_{\beta}$ such that

$$\begin{split} H &= dB_{\alpha} \text{ on } \mathsf{U}_{\alpha}, \\ B_{\beta} - B_{\alpha} &= F_{\alpha\beta} = \left(\nabla^{L_{\alpha\beta}}\right)^{2} \text{ on } \mathsf{U}_{\alpha} \cap \mathsf{U}_{\beta}, \\ \left(L_{\alpha\beta}, \nabla^{L_{\alpha\beta}}\right) \otimes \left(L_{\beta\gamma}, \nabla^{L_{\beta\gamma}}\right) \otimes \left(L_{\gamma\alpha}, \nabla^{L_{\gamma\alpha}}\right) \simeq (\mathbb{C}, d) \text{ on } \mathsf{U}_{\alpha} \cap \mathsf{U}_{\beta} \cap \mathsf{U}_{\gamma}. \end{split}$$

Let \mathcal{L} be the holonomy line bundle on LM arising from H.

Let $\tau \in \mathbb{H}$, the upper half plane. This means now we consider the complex structures on the source $T^2 \to M$.

Roughly speaking, our construction of the τ -average holonomy line bundle is to make sense of the following line bundle over LLM

$$\pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L})^{\otimes \tau},$$

i.e. $\pi_1^*(\mathcal{L}) \otimes$ tensored with the " τ -th power" of $\pi_2^*(\mathcal{L})^{\otimes \tau}$, similar to $a + b\tau \in \mathbb{C}$.

Unfortunately, since τ is not an integer, but a complex number with positive imaginary part, $\pi_2^*(\mathcal{L})^{\otimes \tau}$ does not make sense.

In the following, we will explain a situation that can make sense out of this. Let ξ be a complex line bundle over a manifold X.

Let $\mathfrak{U} = \{U_{\alpha}\}$ be an open good cover of X. Let $\{g_{\alpha\beta}\}$ be a system of U(1)-valued transition functions w.r.t \mathfrak{U} . This gives us a closed Cech cocycle $\{\theta_{\alpha\beta}\}$ valued in \mathbb{R}/\mathbb{Z} by taking $\theta_{\alpha\beta} = \frac{1}{2\pi i} \ln g_{\alpha\beta}$ in the argument interval $[0, 2\pi)$. So $\{\theta_{\alpha\beta}\} \in C^1(\mathfrak{U}, \mathbb{R}/\mathbb{Z})$.

Let

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

be the obvious exact sequence. It is not hard to show that

- (1) $\{\theta_{\alpha\beta}\}\$ can lifted as the image of a Cech cocyle in $C^1(\mathfrak{U}, \mathbb{R}) \iff \xi$ is trivial.
- (2) Different liftings differ by δc for some $c \in C^1(\mathfrak{U}, \mathbb{Z})$.

Let ξ be trivial and $\eta_{\alpha\beta} \in C^1(\mathfrak{U}, \mathbb{R})$ be a lifting of $\{\theta_{\alpha\beta}\}$. Then we can consider the \mathbb{C} -valued functions $\{e^{2\pi i \tau \eta_{\alpha\beta}}\}$, which satisfying

$$e^{2\pi i \tau \eta_{\alpha\beta}} \cdot e^{2\pi i \tau \eta_{\beta\gamma}} \cdot e^{2\pi i \tau \eta_{\gamma\alpha}} = 1$$

and therefore glues us a complex line bundle over X. One can consider it as the τ -th power of ξ . One may think of this construction by taking $\tau = \frac{1}{n}$, $n \in \mathbb{Z}$ and the construction of an n-th root of ξ .

Let us come back to the double loop space. Although $\pi_2^*(\mathcal{L})^{\otimes \tau}$ does not make sense, but suppose

$$p: \mathcal{S} \to LLM$$

be the circle bundle of $\pi_2^*(\mathcal{L})$, the pull back $p^*(\pi_2^*(\mathcal{L}))$ is a trivial bundle over \mathcal{S} . Then $p^*(\pi_2^*(\mathcal{L}))^{\otimes \tau}$ makes sense as explained above. So on \mathcal{S} , we have the line bundle

$$p^*(\pi_1^*(\mathcal{L})) \otimes p^*(\pi_2^*(\mathcal{L}))^{\otimes \tau}.$$

This is just the τ -average holonomy line bundle we are going to construct.

Note that on LLM, there is the T^2 -action. Hence we have to give a T^2 -invariant transition function for the line bundle

$$p^*(\pi_1^*(\mathcal{L})) \otimes p^*(\pi_2^*(\mathcal{L}))^{\otimes \tau}.$$

To achieve this, we have to give $(K_1 + \tau K_2)$ -invariant transition functions for this line bundle, which we will explain in the following.

For any double loop $x \in LLU_{\alpha} \cap LLU_{\beta}$, i.e. $x : T^2 \to U_{\alpha\beta}$, denote the holonomy of the $\nabla^{L_{\alpha\beta}}$ along the K_1 -direction of by hol^1 , which is a function of t; and the holonomy of the $\nabla^{L_{\alpha\beta}}$ along the K_2 -direction by hol^2 , which is a function of s.

Consider the function on $LLU_{\alpha} \cap LLU_{\beta}$

$$g_{\alpha\beta} := e^{\overline{\ln hol_{\alpha_{\beta}}^{1}}^{2}} \cdot e^{\tau \overline{\ln hol_{\alpha_{\beta}}^{2}}^{1}}.$$

Note here for $\ln hol^1$, it is continuously defined for $t \in [0, 1)$, and for $\ln hol^2$, it is continuously defined for $s \in [0, 1)$.

$$\begin{split} &L_{K_1+\tau K_2} g_{\alpha\beta} \\ =& L_{K_1+\tau K_2} \left(e^{\overline{\ln hol_{\alpha_{\beta}}^{1}}^2} \cdot e^{\tau \overline{\ln hol^2}_{\alpha_{\beta}}^1} \right) \\ =& e^{\overline{\ln hol_{\alpha_{\beta}}^{1}}^2} \cdot e^{\tau \overline{\ln hol^2}_{\alpha_{\beta}}^1} [L_{\tau K_2} \overline{\ln hol_{\alpha_{\beta}}^1}^2 + L_{K_1} \tau \overline{\ln hol^2}_{\alpha_{\beta}}^1] \\ =& e^{\overline{\ln hol_{\alpha_{\beta}}^{1}}^2} \cdot e^{\tau \overline{\ln hol^2}_{\alpha_{\beta}}^1} [\tau \overline{K_2 \ln hol_{\alpha_{\beta}}^1}^2 + \tau \overline{K_1 \ln hol^2}_{\alpha_{\beta}}^1] \\ =& e^{\overline{\ln hol_{\alpha_{\beta}}^{1}}^2} \cdot e^{\tau \overline{\ln hol^2}_{\alpha_{\beta}}^1} \tau [\overline{\iota_{K_2} d \ln hol_{\alpha_{\beta}}^1}^2 + \overline{\iota_{K_1} d \ln hol^2}_{\alpha_{\beta}}^1] \\ =& e^{\overline{\ln hol_{\alpha_{\beta}}^{1}}^2} \cdot e^{\tau \overline{\ln hol^2}_{\alpha_{\beta}}^1} \tau \left[\overline{\iota_{K_2} \iota_{K_1} \overline{F_{\alpha_{\beta}}}^1}^2 + \overline{\iota_{K_1} \iota_{K_2} \overline{F_{\alpha_{\beta}}}^2}^1 \right] \\ =& e^{\overline{\ln hol_{\alpha_{\beta}}^{1}}^2} \cdot e^{\tau \overline{\ln hol^2}_{\alpha_{\beta}}^1} \tau \left[\iota_{K_1} \iota_{K_2} \overline{\overline{F_{\alpha_{\beta}}}} + \iota_{K_2} \iota_{K_1} \overline{\overline{F_{\alpha_{\beta}}}} \right] \\ =& 0. \end{split}$$

So $g_{\alpha\beta}$ is $(K_1 + \tau K_2)$ -invariant.



Denote

$$h_{\alpha\beta\gamma} := g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}.$$

On the triple intersection $LLU_{\alpha} \cap LLU_{\beta} \cap LLU_{\gamma}$,

$$hol_{\alpha\beta}^{i}hol_{\beta\gamma}^{i}hol_{\gamma\alpha}^{i}=1,\ i=1,2.$$

Hence $\ln hol_{\alpha\beta}^i + \ln hol_{\beta\gamma}^i + \ln hol_{\gamma\alpha}^i \in 2\pi i\mathbb{Z}$. As $\ln hol$'s are continuously defined, one must have

$$h_{\alpha\beta\gamma} = e^{2\pi i m_{\alpha\beta\gamma}\tau}$$

for some $m_{\alpha\beta\gamma} \in \mathbb{Z}$, where $\{m_{\alpha\beta\gamma}\}$ forms the Cech cocycle representing $\pi_2^*(c_1(\mathcal{L}_B))$ in $H^2(LLM,\mathbb{Z})$ with $c_1(\mathcal{L}_B)$ being the first Chern class of the holonomy line bundle \mathcal{L}_B on LM arising from the ω on M.

Let $p: \mathcal{S}_B \to LM$ be the circle bundle of the line bundle $\mathcal{L}_B \to LM$. Then $p^*\mathcal{L}_B$ is a trivial line bundle over \mathcal{S}_B . Hence the class $p^*(c_1(\mathcal{L}_B))$ be 0 on \mathcal{S}_B . Therefore $\widetilde{\pi}_2^* \circ p^*(c_1(\mathcal{L}_B))$ is 0 on $p^*\mathcal{S}_B$, the pulled back circle bundle over LLM.

$$(4) p^* \mathcal{S}_B \xrightarrow{\widetilde{\pi}_2} \mathcal{S}_B$$

$$\downarrow p$$

$$\downarrow p$$

$$\downarrow LLM \xrightarrow{\pi_2} LM$$

For simplicity, in the sequel we will denote the total space p^*S_B by LLM^{ω} , which carries the induced T^2 -action arising from LLM.

Using $\{g_{\alpha\beta}\}$, after some treatment on LLM^{ω} , we can construct a complex line bundle $\mathcal{L}_{B,\tau}$ on LLM^{ω} , which we call average τ -holonomy line bundle, which also carries a canonical connection $\nabla^{\mathcal{L}_{B,\tau}}$ out from the B-field.

D. Equivariantly super flatness

Denote by $\Omega_{bas}^{\bullet}(LLM^{\omega}, \mathcal{L}_{H,\tau})$ the space of basic differential forms on LLM^{ω} with values in the average τ -holonomy line bundle $\mathcal{L}_{B,\tau}$. Here basic form means that contracted with vertical tangent vectors gives 0. Let u be an indeterminate such that $\deg u = 2$. Consider the odd operator

(5)
$$Q_{B,\tau} := \nabla^{\mathcal{L}_{B,\tau}} - w_{K_1 + \tau K_2} + u^{-1} \widetilde{p}^* \overline{\overline{\omega}}$$

which acts on $\Omega^{\bullet}(LLM^{\omega}, \mathcal{L}_{B,\tau})_{bas}[[u, u^{-1}]].$

Theorem (H.-Mathai)

The following identities hold,

$$\frac{1}{2}[Q_{B,\tau},Q_{B,\tau}] = Q_{B,\tau}^2 = -uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}, \quad [Q_{B,\tau},-uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}] = 0,$$

where $L_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}$ is the Lie derivative along the direction $K_1+\tau K_2$.

So the odd operator $Q=Q_{B,\tau}$ and the even operator $P=-uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}$ obey the relations

$$\frac{1}{2}[Q, Q] = P, \quad [Q, P] = 0$$

of the superalgebra considered in Witten's paper Supersymmetry and Morse theory.

E. Localization

The above theorem tells us there is a complex

$$\left(\Omega_{bas}^{\bullet}(LLM^{\omega},\mathcal{L}_{B,\tau})^{K_1+\tau K_2}[[u,u^{-1}]],Q_{B,\tau}\right).$$

Note that the zeros of the complex vector field $K_1 + \tau K_2$ are T^2 -fixed points of LLM, i.e. M. We have the Borel-Witten type localization :

Theorem (H.-Mathai)

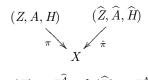
Let $i: M \times S^1 \to LLM^{\omega}$ be the inclusion map. Then the restriction map

$$i^*: \left(\Omega_{bas}^{\bullet}(LLM^{\omega}, \mathcal{L}_{B,\tau})^{K_1 + \tau K_2}[[u, u^{-1}]], Q_{B,\tau}\right) \\ \to \left(\Omega_{bas}^{\bullet}(M \times S^1)[[u, u^{-1}]], d + u^{-1}p^*\omega\right) \cong \left(\Omega^{\bullet}(M)[[u, u^{-1}]], d + u^{-1}\omega\right)$$

is a quasi-isomorphism, $\forall \tau \in \mathbb{H}$.

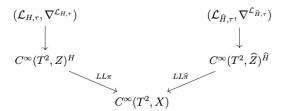
Now we apply our theory on double loop spaces to study T-duality from the perspective of 2d

Recall that we have the T-dual pair with H-flux:



 $\pi_!(H) = F^{\widehat{A}}, \ \ \widehat{\pi}_!(\widehat{H}) = F^A$

Double loop the T-dual pair, we have the following picture:



A sheaf over upper half plane \mathbb{H}

For the pair (Z, H), define a sheaf $(G(C^{\infty}(T^2, Z)^H, \mathcal{L}_H), \mathcal{Q}_H)$ on \mathbb{H} of commutative differential graded algebras that to $U \subset \mathbb{H}$ assigns the graded complex of $\mathcal{O}(U)$ -modules

$$(G(C^{\infty}(T^2, Z)^H, \mathcal{L}_H)(U), \mathcal{Q}_H)$$

$$:= \bigoplus_{m \in \mathbb{Z}} \left(\mathcal{O}(U; \Omega_{bas}^{\bullet, \mathbb{T}}(C^{\infty}(T^2, Z)^H, \mathcal{L}_{H, \tau}^{\otimes m})[[u, u^{-1}]]^{K_1 + \tau K_2}) \cdot y^m, \ Q_{mB, \tau} \right),$$

where $\mathcal{O}(U; \Omega_{bas}^{\bullet, \mathbb{T}}(C^{\infty}(T^2, Z)^H, \mathcal{L}_{H, \tau}^{\otimes m})[[u, u^{-1}]]^{K_1 + \tau K_2})$ means i.e for each $\tau \in U$, one assigns to it an element in $\Omega_{bas}^{\bullet, \mathbb{T}}(C^{\infty}(T^2, Z)^H, \mathcal{L}_{H, \tau}^{\otimes m})[[u, u^{-1}]]^{K_1 + \tau K_2}$. Dually, one can also define the sheaf $(G(C^{\infty}(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}), \mathcal{Q}_{\widehat{H}})$.

Note that we have assembled mH for $m \in \mathbb{Z}$, i.e. we consider the graded version. Here y is a formal variable to keep track of the level m by y^m .

Passing to cohomology, we get the sheaves $\mathcal{G}(C^{\infty}(T^2, Z)^H, \mathcal{L}_H)$ and $\mathcal{G}(C^{\infty}(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}})$. The localisation theorem tells us that the restriction maps

$$res: \mathcal{G}(C^{\infty}(T^2, Z)^H, \mathcal{L}_H) \to \mathcal{G}(Z, H), \quad \widehat{res}: \mathcal{G}(C^{\infty}(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}) \to \mathcal{G}(\widehat{Z}, \widehat{H})$$
 are isomorphisms of sheaves.

In a previous work joint with Mathai, we constructed the **graded Hori morphisms** between the sheaves

$$GHor_*: (G(Z, H), D^H) \to (G(\widehat{Z}, \widehat{H}), D^{\widehat{H}}), \ GHor: \mathcal{G}(Z, H) \to \mathcal{G}(\widehat{Z}, \widehat{H}).$$

$$\widehat{GHor}_*: (\mathrm{G}(\widehat{Z},\widehat{H}), D^{\widehat{H}}) \to (\mathrm{G}(Z,H), D^H), \ \widehat{GHor}: \mathcal{G}(\widehat{Z},\widehat{H}) \to \mathcal{G}(Z,H).$$

and showed that they send Jacobi forms to Jacobi forms.



Now we construct the graded Hori morphisms for 2d σ -models by

$$\begin{split} &\mathit{GHor}^{\sigma} := \widehat{\mathit{res}}^{-1} \circ \mathit{GHor} \circ \mathit{res} : \mathcal{G}(\mathit{C}^{\infty}(\mathit{T}^{2}, \mathit{Z})^{\mathit{H}}, \mathcal{L}_{\mathit{H}}) \to \mathcal{G}(\mathit{C}^{\infty}(\mathit{T}^{2}, \widehat{\mathit{Z}})^{\widehat{\mathit{H}}}, \mathcal{L}_{\widehat{\mathit{H}}}), \\ &\widehat{\mathit{GHor}}^{\sigma} := \mathit{res}^{-1} \circ \widehat{\mathit{GHor}} \circ \widehat{\mathit{res}} : \mathcal{G}(\mathit{C}^{\infty}(\mathit{T}^{2}, \widehat{\mathit{Z}})^{\widehat{\mathit{H}}}, \mathcal{L}_{\widehat{\mathit{H}}}) \to \mathcal{G}(\mathit{C}^{\infty}(\mathit{T}^{2}, \mathit{Z})^{\mathit{H}}, \mathcal{L}_{\mathit{H}}), \\ & \text{assembled in the following commutative diagram,} \end{split}$$

$$\mathcal{G}(C^{\infty}(T^{2},Z)^{H},\mathcal{L}_{H}) \xleftarrow{GHor^{\sigma}} \mathcal{G}(C^{\infty}(T^{2},\widehat{Z})^{\widehat{H}},\mathcal{L}_{\widehat{H}})$$

$$\downarrow^{res} \cong \qquad \qquad \downarrow^{\widehat{res}} \cong$$

$$\mathcal{G}(Z,H) \xleftarrow{GHor} \mathcal{G}(\widehat{Z},\widehat{H})$$

Theorem (H.-Mathai)

One has

(6)
$$\widehat{GHor}^{\sigma} \circ GHor^{\sigma} = -y \frac{\partial}{\partial y}, \quad GHor^{\sigma} \circ \widehat{GHor}^{\sigma} = -y \frac{\partial}{\partial y}.$$



References

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- 2. A. Strominger, S.T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Physics B Volume 479, Issues 1–2, 11 November 1996, Pages 243-259.
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Happy Birthday, Uli!



