Real Spin Bordism & Orientations of Topological K-Theory

Yigal Kamel

(joint work with Zach Halladay)

CLASSICAL ORIENTATIONS OF K-THEORY

$$\boxed{\begin{array}{c}\text{MAPS OF}\\ \text{RING SPECTRA}\\ MG \to E\end{array}} \quad \leftrightsquigarrow \quad \boxed{\begin{array}{c}\text{THOM ISOMORPHISMS}\\ \text{IN } E\text{-COHOMOLOGY}\\ \text{FOR } G\text{-VECTOR BUNDLES}\end{array}}$$

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$$MSpin \to KO$$

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$\mathbb{R}^{p,q}$

C_2-representation
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Theorem (Atiyah): $G \longrightarrow \mathrm{Spin}^c(p,q)$ C_2-equivariant

$$\Rightarrow KU_{\mathbb{R},G}(X) \xrightarrow[\cdot u]{\sim} KU_{\mathbb{R},G}(\mathbb{R}^{p,q} \times X), \quad p \equiv q \bmod 8.$$

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↳ contains all Thom isomorphisms above,

e.g. $\mathrm{Spin}(8n) \hookrightarrow \mathrm{Spin}^c(8n) = \mathrm{Spin}^c(8n,0)$

& $U(n) \hookrightarrow \mathrm{Spin}^c(n,n)$ C_2-equivariantly

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Real spin orientation

(1)+(3) $\Longrightarrow$ $MSpin^c \longrightarrow KU$

(2)+(3) $\Longrightarrow$ $MSpin \longrightarrow KO$

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THE FIXED POINTS OF $MSpin^c_{\mathbb{R}}$

Recall:

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Spin(n) \arrow[rr] \arrow[dr, hook] & & Spin^c(n) \\
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$\hookrightarrow$ Actually shows $\left(\mathrm{MSpin} \xrightarrow{u} \mathrm{MSpin}^c\right) \not\simeq \left(E^{C_2} \to E^e\right) \ \forall E$

THANKS FOR LISTENING!

OVERVIEW

$$
\begin{tikzcd}
 & MSpin \arrow[r] \arrow[d, "\cancel{\wr}"] & KO \arrow[d, "\wr"] \\
 & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[r] \arrow[d] & (KU_{\mathbb{R}})^{C_2} \arrow[d] \\
MU_{\mathbb{R}} \arrow[r] & MSpin^c_{\mathbb{R}} \arrow[r] & KU_{\mathbb{R}}
\end{tikzcd}
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