Stable equivalence relations of 4-manifolds

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May 17, 2024

arXiv:2405.06637

Stable equivalence relations

All manifolds are smooth, orientable, closed and connected.

An equivalence relation $\sim$ is stable if

$$M \sim M\#(S^n \times S^n)$$

Examples:

- Stable diffeomorphism in dim 4
  $\Updownarrow$
- Stable homeomorphism
- Homotopy equivalence up to stabilisations
- Existence of K, K' 1-connected such that $M\#K \cong M'\#K'$
- (Immersion equivalence ($S^{2n} \bowtie S^n \times S^n$))

Stable diffeomorphism

$\pi := \pi_1 M$

$$\begin{tikzcd} & B \arrow[d, "\xi"] & (n-1)\text{-type} \\ M \arrow[ur, "\tilde{\nu}\ \text{(n-1)-smoothing}"] \arrow[r, "\nu"] & BSO & \end{tikzcd}$$

The normal $(n-1)$-type of M is an n-coconnected map $\xi\colon B \to BSO$ such that there exists an n-connected map $\widetilde{\nu}\colon M \to B$ lifting the stable normal bundle $\nu\colon M \to BSO$. Any such lift $\widetilde{\nu}$ is called a normal $(n-1)$-smoothing.

Theorem (Kreck)

Two $2n$-manifolds are stably diffeomorphic if they have the same normal $(n-1)$-type and admit bordant normal $(n-1)$-smoothings.

$$\left\{ \begin{array}{l} 2n\text{-manifolds with} \\ (n-1)\text{-type } \xi \end{array} \right\} \Big/ \text{stable diffeo} \cong \Omega_{2n}(\xi) \big/ \mathrm{Aut}(\xi)$$

M^4 $\quad w_2(\widetilde{M}) \neq 0$

$$\begin{tikzcd} & B\pi \times BSO \arrow[d, "pr_2"] \\ M \arrow[ur, "{(c,\nu)}"] \arrow[r] & BSO \end{tikzcd}$$

$$\Omega_4^{SO}(B\pi) \big/ \mathrm{Out}(\pi)$$

Theorem (Kreck)

Two $2n$-manifolds are stably diffeomorphic if they have the same normal $(n-1)$-type and admit bordant normal $(n-1)$-smoothings.

In particular, every $A \leq \Omega_{2n}(\xi)$ gives rise to a stable equivalence relation. Is there a geometric meaning of these equivalence relations?

Given a stable equivalence relation that implies that M and M' have the same normal $(n-1)$-type ξ, is there a subgroup A of $\Omega_{2n}(\xi)$ such the relation holds if and only if there are normal $(n-1)$-smoothings $\widetilde{\nu}, \widetilde{\nu}'$ such that

$$[M, \widetilde{\nu}] - [M', \widetilde{\nu}'] \in A \leq \Omega_4(\xi)?$$

Normal 1-types of 4-manifolds

- $\pi := \pi_1(M)$
- If $\widetilde{M}$ is not spin, then the normal 1-type of M is $\mathrm{pr}_2 \colon B\pi \times BSO \to BSO$.
- If M is spin, the normal 1-type of M is $B\pi \times BSpin \to BSO$, where the map is the projection to $BSpin$ followed by the canonical map to BSO.
- If $\widetilde{M}$ is spin, but M is not, the normal 1-type is a twisted version of $B\pi \times BSpin \to BSO$.

$$c^* w = w_2(M)$$

$$\begin{tikzcd} BSpin \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "\xi"'] & B\pi \arrow[d, "w"] \\ BSpin \arrow[r] & BSO \arrow[r] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

The totally non-spin case

- Assume $\widetilde{M}$ is not spin.
- We have
$$\Omega_4^{SO}(B\pi) \cong \mathbb{Z} \oplus H_4(\pi;\mathbb{Z}).$$
$$[c\colon M \to B\pi] \mapsto (\sigma(M), c_*[M])$$
- The signature and $c_*[M]$ are homotopy invariants. Hence homotopy equivalence up to stabilisations is the same as stable diffeomorphism.
- There exist K, K' simply connected such that

$$M\#K \cong M'\#K'$$

Add $\mathbb{CP}^2$'s to make sign the same

if and only if $c_*[M] = c'_*[M] \in H_4(\pi;\mathbb{Z})/\operatorname{Out}(\pi)$.

The spin case

In the spin case we have to understand $\Omega_4^{spin}(B\pi)$.

n	0	1	2	3	4
Ω_n^{spin}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$16\mathbb{Z}$

AHSS $H_p(\pi; \Omega_q^{spin}) \Rightarrow \Omega_{p+q}^{spin}(B\pi)$

Yields a filtration

$$0 \subseteq F_{0,4} \subseteq F_{2,2} \subseteq F_{3,1} \subseteq \Omega_4^{spin}(B\pi)$$

$$F_{0,4} = \Omega_4^{spin} \overset{\text{sign}}{=} 16\mathbb{Z}$$

Stable equivalence relations of spin 4-manifolds

algebraic	geometric
0	stable diffeo
$F_{0,4}$	$\exists L, L'$ 1-conn, spin s.t. $M \# L \cong M' \# L'$
$F_{2,2}$	?
$F_{3,1}$	$\exists K, K'$ 1-conn s.t. $M \# K \cong M' \# K'$
?	h.e. up to $\#(S^2 \times S^2)$'s

Classical surgery

- $\mathcal{S}(M) = \{f\colon N \xrightarrow{\simeq} M\}/$ diffeomorphism
- $\mathcal{N}(M) =$ degree 1 normal maps up to normal bordism.
- The surgery exact sequence

Something algebraic $\rightarrow$

$$L_5(\mathbb{Z}\pi) \to \mathcal{S}(M) \to \mathcal{N}(M) \xrightarrow{\alpha} L_4(\mathbb{Z}\pi)$$

This is not exact.

It is exact "stably".

$$\mathcal{N}(M) \cong [M, G/TOP] \cong \mathbb{Z} \oplus H_2(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z} \oplus H_2(\pi;\mathbb{Z}/2)$$

sign

K_2

$$\begin{array}{ccccc} \mathcal{N}(M) & \longrightarrow & \Omega_4^{TopSpin}(M) & \longrightarrow & \Omega_4^{TopSpin}(B\pi) \\ & & \cup & & \cup \\ f\colon N \to M & \longmapsto & [N,f]-[M,id] \in F_{2,2}^M & \longrightarrow & F_{2,2} \end{array}$$

$\cong$

Theorem (K-Nicholson-Veselá)

There exist normal 1-smoothings such that $[M] - [M'] \in F_{2,2}$ *if and only if there exists* k' *such that there is a degree one normal map*

$$M'\#k'(S^2 \times S^2) \to M. \# k(S^2 \times S^2)$$

$$\mathcal{M}$$

Theorem (K-Nicholson-Veselá)

M and M' are homotopy equivalent up to stabilisations if and only if there exist normal 1-smoothings such that

$$[M] - [M'] \in \overline{\ker \kappa_2 \cap \ker(w \cap -)} \leq F_{2,2} \leq \Omega_4^{spin}(B\pi).$$

Stable equivalence relations

$w \in H^2(\pi; \mathbb{Z}/2)$

$\downarrow$

$w_2(M)\ H^2(M; \mathbb{Z}/2)$

algebraic	geometric
0	stable diffeomorphism
$F_{0,4}$	There exist 1-connected spin manifolds L, L' such that $M \# L \cong M' \# L'$.
$[\ker \kappa_2 \cancel{\cap \ker(w \cap \quad)}]$	homotopy equivalence up to stabilisations
$F_{2,2}$	There exist k' and a degree 1 normal map $M' \# k'(S^2 \times S^2) \to M$
$F_{3,1}$	There exist 1-connected manifolds K, K' such that $M \# K \cong M' \# K'$.

Stable rigidity

- We say that a group π satisfies stable rigidity if homotopy equivalent 4-manifolds with fundamental group π are stably diffeomorphic.
- (Davis) If κ_2 is injective for π, then π satisfies stable rigidity.
- If π is torsion-free and satisfies the Farrell–Jones conjecture, κ_2 is injective.
- (Teichner) D_∞ and Q_{4n} for $n \geq 3$ do not satisfy stable rigidity.

Theorem (K–Nicholson–Veselá)

All abelian groups satisfy stable rigidity.

Theorem (K–Nicholson–Veselá)

A finite group π with dihedral Sylow 2-subgroup satisfies stable rigidity if and only if π is not a semi-direct product $P \rtimes D_{2^n}$ with $n \geq 3$.

Open questions

Question

Are there homotopy equivalent spin 4-manifolds that are not stably diffeomorphic?

In that case the fundamental group cannot be abelian, quaternion, dihedral or semi-dihedral. By the Farrell–Jones conjecture, it probably cannot be torsion-free.

Question

Is there a group π such that $\ker \kappa_2^h \neq \ker \kappa_2^s$?

This is equivalent to the existence of homotopy equivalent 4-manifolds that are not simple homotopy equivalent up to stabilisations.

Question

What about $2n$-manifolds for $n \geq 3$?