Stable equivalence relations of 4-manifolds

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May 17, 2024

arXiv:2405.06637

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Stable equivalence relations

All manifolds are smooth, orientable, closed and connected.

An equivalence relation \sim is stable if

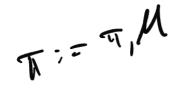
 $M \sim M \# (S^n \times S^n)$

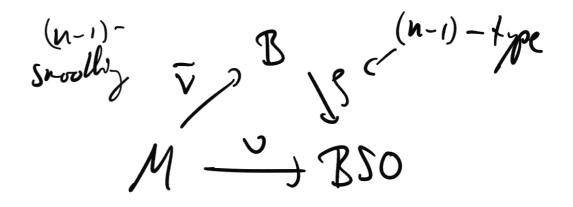
Examples:

- Stable diffeomorphism is den 4
- Stable homeomorphism
- Homotopy equivalence up to stabilisations
- Existence of K, K' 1-connected such that $M \# K \cong M' \# K'$ (• Immersion equivalence $(\mathcal{J} \bowtie \mathcal{J} \mathcal{J}))$

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Stable diffeomorphism

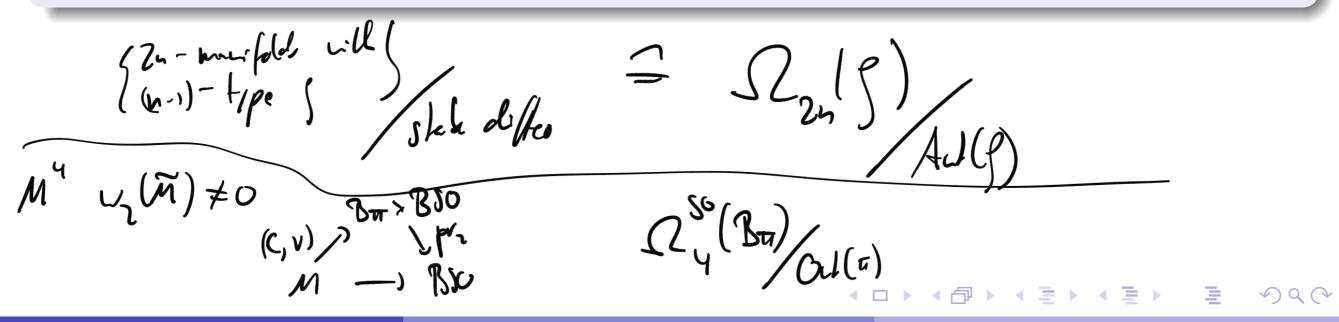




The normal (n-1)-type of M is an n-coconnected map $\xi \colon B \to BSO$ such that there exists an n-connected map $\tilde{\nu} \colon M \to B$ lifting the stable normal bundle $\nu \colon M \to BSO$. Any such lift $\tilde{\nu}$ is called a normal (n-1)-smoothing.

Theorem (Kreck)

Two 2n-manifolds are stably diffeomorphic if they have the same normal (n - 1)-type and admit bordant normal (n - 1)-smoothings.



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Theorem (Kreck)

Two 2n-manifolds are stably diffeomorphic if they have the same normal (n - 1)-type and admit bordant normal (n - 1)-smoothings.

In particular, every $A \leq \Omega_{2n}(\xi)$ gives rise to a stable equivalence relation. Is there a geometric meaning of these equivalence relations?

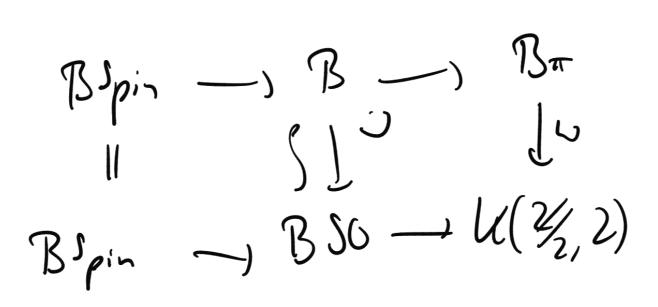
Given a stable equivalence relation that implies that M and M' have the same normal (n-1)-type ξ , is there a subgroup A of $\Omega_{2n}(\xi)$ such the relation holds if and only if there are normal (n-1)-smoothings $\tilde{\nu}, \tilde{\nu}'$ such that

$$[M,\widetilde{\nu}]-[M',\widetilde{\nu}']\in A\leq \Omega_4(\xi)?$$

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Normal 1-types of 4-manifolds

- $\pi := \pi_1(M)$
- If *M* is not spin, then the normal 1-type of *M* is $pr_2: B\pi \times BSO \rightarrow BSO$.
- If M is spin, the normal 1-type of M is Bπ × BSpin → BSO, where the map is the projection to BSpin followed by the canonical map to BSO.
- If M is spin, but M is not, the normal 1-type is a twisted version of $B\pi \times BSpin \rightarrow BSO$.



$$C^{*} U = W(M)$$

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The totally non-spin case

• Assume \widetilde{M} is not spin.

• We have

$$\Omega_4^{SO}(B\pi) \cong \mathbb{Z} \oplus H_4(\pi;\mathbb{Z}).$$

[c: $M \to B\pi$] $\mapsto (\sigma(M), c_*[M])$

- The signature and c_{*}[M] are homotopy invariants. Hence homotopy equivalence up to stabilisations is the same as stable diffeomorphism.
- There exist K, K' simply connected such that

 $M \# K \cong M' \# K'$

Adl P's to mke sign the same

if and only if $c_*[M] = c'_*[M] \in H_4(\pi; \mathbb{Z}) / \operatorname{Out}(\pi)$.

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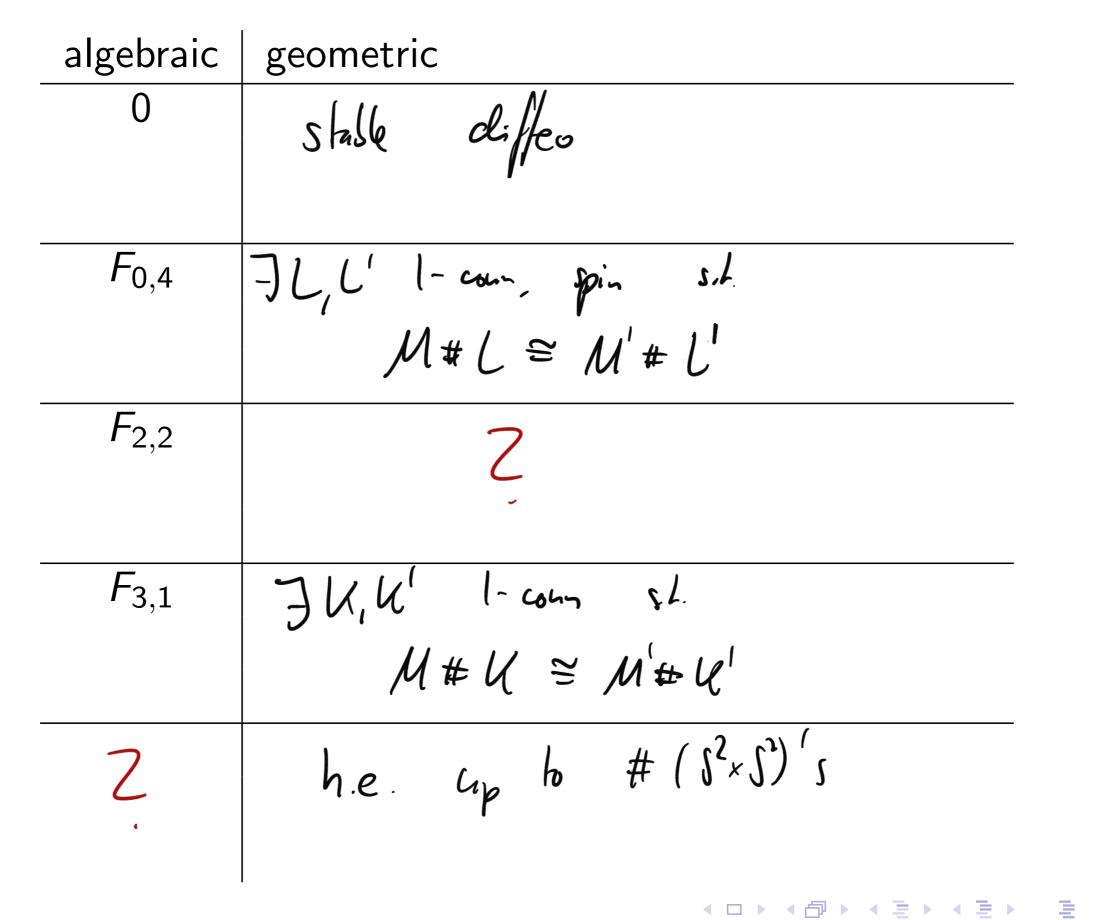
The spin case

In the spin case we have to understand $\Omega_4^{spin}(B\pi)$.

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Stable equivalence relations of spin 4-manifolds



Classical surgery

- $\mathcal{S}(M) = \{f : N \xrightarrow{\simeq} M\} / \text{ diffeomorphism }$
- $\mathcal{N}(M) = \text{degree 1 normal maps up to normal bordism.}$
- The surgery exact sequence

Schelly
$$L_5(\mathbb{Z}\pi) \to S(M) \to \mathcal{N}(M) \xrightarrow{\alpha} L_4(\mathbb{Z}\pi)$$

algebraic $L_5(\mathbb{Z}\pi) \to S(M) \to \mathcal{N}(M) \xrightarrow{\alpha} L_4(\mathbb{Z}\pi)$
This is not exact. $[\mathcal{M}, \mathcal{G}'_{\mathsf{TOP}}] \xrightarrow{sis} \mathcal{K}_1$
If is exact $skilly'' = [\mathcal{M}, \mathcal{G}'_{\mathsf{TOP}}] \to \mathcal{I} \oplus H_1(\pi; \mathcal{H}_1)$
 $\mathcal{I} \oplus H_1(\mathcal{H}; \mathcal{H}_1) \to \mathcal{I} \oplus H_1(\pi; \mathcal{H}_1)$

$$N(M) \longrightarrow \mathcal{L}_{Y}(M) \longrightarrow \mathcal{L}_{Y}(R_{\pi})$$

$$F(M) \longrightarrow [M] - [M] \in F_{2,2}$$

$$F_{2,2}$$

Theorem (K-Nicholson-Veselá)

There exist normal 1-smoothings such that $[M] - [M'] \in F_{2,2}$ if and only if there exists k' such that there is a degree one normal map

$$M' \# k'(S^2 \times S^2) \rightarrow M. \# k(J^* J^*)$$

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Theorem (K-Nicholson-Veselá)

M and M' are homotopy equivalent up to stabilisations if and only if there exist normal 1-smoothings such that

 $[M] - [M'] \in \overline{\ker \kappa_2 \cap \ker(w \cap -)} \leq F_{2,2} \leq \Omega_4^{spin}(B\pi).$

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Stable equivalence relations

able equivalence relations $\bigvee \in H^2(\pi; \mathcal{X})$	
	L
	رر) ۲ ^۲ (۳, ¼) geometric
algebraic	geometric
0	stable diffeomorphism
F _{0,4}	There exist 1-connected spin manifolds L, L'
	such that $M \# L \cong M' \# L'$.
[ker $\kappa_2 \cap \text{ker}(w \cap -)$]	homotopy equivalence up to stabilisations
<i>F</i> _{2,2}	There exist k' and a degree 1 normal map
	$M' \# k'(S^2 imes S^2) o M$
<i>F</i> _{3,1}	There exist 1-connected manifolds K, K'
	such that $M \# K \cong M' \# K'$.

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Stable rigidity

- We say that a group π satisfies stable rigidity if homotopy equivalent 4-manifolds with fundamental group π are stably diffeomorphic.
- (Davis) If κ_2 is injective for π , then π satisfies stable rigidity.
- If π is torsion-free and satisfies the Farrell–Jones conjecture, κ_2 is injective.
- (Teichner) D_{∞} and Q_{4n} for $n \geq 3$ do not satisfy stable rigidity.

Theorem (K–Nicholson–Veselá)

All abelian groups satisfy stable rigidity.

Theorem (K–Nicholson–Veselá)

A finite group π with dihedral Sylow 2-subgroup satisfies stable rigidity if and only if π is not a semi-direct product $P \rtimes D_{2^n}$ with $n \ge 3$.

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Open questions

Question

Are there homotopy equivalent spin 4-manifolds that are not stably diffeomorphic?

In that case the fundamental group cannot be abelian, quaternion, dihedral or semi-dihedral. By the Farrell–Jones conjecture, it probably cannot be torsion-free.

Question

Is there a group π such that ker $\kappa_2^h \neq \ker \kappa_2^s$?

This is equivalent to the existence of homotopy equivalent 4-manifolds that are not simple homotopy equivalent up to stabilisations.

Question

What about 2n-manifolds for $n \geq 3$?

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