Homology Spheres, Acyclic Groups and Kan-Thurston Theorem

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From Analysis to Homotopy Theory

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Basic Notions

Definitions

1. An *integer homology sphere* is a smooth closed n-manifold Σ^n such that
$$H_*(\Sigma^n) \cong H_*(\mathbb{S}^n).$$
2. A discrete group G is called *perfect* if $H_1(G) = 0$.
3. A discrete group G is called *superperfect* if $H_1(G) = H_2(G) = 0$.

E.g., $\forall n \geqslant 3$ the group $\mathrm{SL}(n, \mathbb{F}_q)$ is superperfect except for $\mathrm{SL}(3, \mathbb{F}_2)$, $\mathrm{SL}(4, \mathbb{F}_2)$, $\mathrm{SL}(3, \mathbb{F}_4)$.

Kervaire and Novikov's Result

Proposition (folklore)

Let Σ^n be a homology n-sphere ($n \geqslant 3$) with fundamental group G. Then G is superperfect.

Theorem (M. Kervaire'69, S. Novikov'62 (unpubl.))

If G is a finitely presented superperfect group then for any $n \geqslant 5$ *there exists* a homology n-sphere Σ^n with $\pi_1(\Sigma^n) = G$.

Poincaré Sphere

Recall, it is

$$\mathbb{S}^3/2I,$$

where

$$2I \cong \langle a, b \mid a^2 = b^3 = (ab)^5 \rangle \cong \mathrm{SL}(2, \mathbb{Z}/5)$$

is called the *binary icosohedral group* acting by some quaternionic representation on $\mathbb{S}^3 \subset \mathbb{H}$ by the left multiplication.

Theorem (J. Milnor'57)

The Poincaré sphere is *the only homology 3-sphere* with a finite nontrivial fundamental group up to homeomorphism.

The Poincaré sphere is the Brieskorn sphere $\Sigma(2, 3, 5)$.

Deficiency of a Group

Definition

The *deficiency* def(G) of a finitely presented group G:

$$\max\{f - r \mid f = |F|, r = |R|, G \cong \langle F \mid R \rangle\}$$

over all representations of G with finite f and r.

E.g., $\text{def}(2I) = 0$, since we have

Proposition (folklore)

If a superperfect group has a balanced representation with $|F| = |R|$, then its deficiency vanishes.

Deficiency of a Group

The Proposition above follows from

Epstein's Lemma

For any finitely presented group $G = \langle F \mid R \rangle$ we have

$$|F| - |R| \leqslant \operatorname{rank} H_1(G) - s(H_2(G)),$$

where $s(H_2(G))$ denotes the minimal number of generators of the group $H_2(G)$.

Thus, we get

Proposition (folklore)

The fundamental groups of any homology 3-sphere has zero deficiency.

Brieskorn Spheres Σ_a

Definition

For positive integers a_1, a_2 and a_3 a *3-dimensional Brieskorn manifold*

$$M(a_1, a_2, a_3) := \{z \in \mathbb{C}^3 \mid z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\} \cap \{z \in \mathbb{C}^3 \mid |z| = 1\}.$$

There is an equivalent characterization as some Seifert manifold (W. Neumann, F. Raymond'77).

M_a is a homology sphere $\Leftrightarrow$ a_1, a_2 and a_3 are pairwise coprime (Brieskorn'66).

A triple $a = (a_1, a_2, a_3)$ is a complete topological invariant for Brieskorn spheres Σ_a (W. Neumann'70).

Fundamental Group of Σ_a

$$\pi_1(\Sigma_a) \cong \Gamma'(a_1, a_2, a_3)$$

$$1 \to C \to \Gamma(p,q,r) \to D(p,q,r) \to 1 - \text{central}$$

$$D(p,q,r) = \langle x, y, z \mid x^p = y^q = z^r = xyz = 1 \rangle$$

This is by J. Milnor'75.

From Seifert's characterization, one gets:

$$\pi_1(\Sigma_a) = \langle x_1, x_2, x_3, h \mid \forall i: \; [h, x_i] = 1, \; x_1 x_2 x_3 = 1, \; x_i^{a_i} h = 1 \rangle$$

From Milnor's results, one can derive

Proposition (K.)

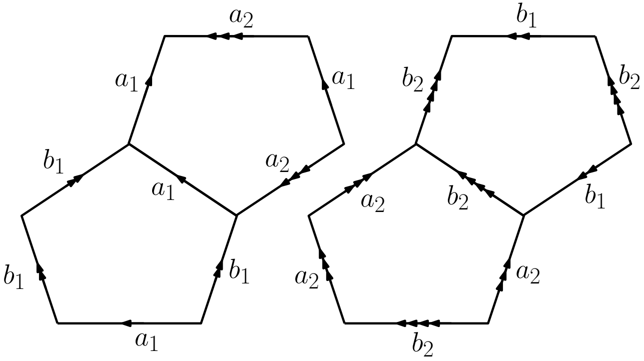
$\Gamma(p,q,r)$ has zero deficiency for any $p, q, r \geqslant 1$.

Acyclic Groups

Definition

A discrete group G is called *acyclic* if

$$H_*(G, \mathbb{Z}) \cong H_*(\{e\}, \mathbb{Z}).$$



$$\mathrm{Hig}_4 = \{a_1, b_1, a_2, b_2 \mid b_1^{a_1} = b_1^2,\ a_1^{a_2} = a_1^2,\ a_2^{b_2} = a_2^2,\ b_2^{b_1} = b_2^2\}$$

Higman Groups

More generally, Higman defined

$$\mathrm{Hig}_n = \langle x_i,\ i \in \mathbb{Z}/n \mid x_i^{x_{i-1}} = x_i^2 \rangle,$$

where $x_i^{x_{i-1}} := x_{i-1}^{-1} x_i x_{i-1}$.

Proposition (N. Monod'17)

Hig_n is acyclic for all $n \geqslant 4$.

Universal Acyclic Groups

Definition

A finitely presented group G is called *universal* if it contains all recursively enumerable finitely generated groups.

The combination of Higman and Chiodo–Hill's results gives

Proposition (K.)

Every finitely presented group can be embedded into a universal acyclic group with 12 generators and 38 relations.

$\pi_1(\Sigma_3)$ Is Not Acyclic

Proposition (A. Berrick, J. Hillman'03)

The fundamental group (nontrivial) of any 3-manifold cannot be acyclic.

Corollary

The groups Hig_n ($n \geqslant 4$) cannot serve as fundamental groups of homology 3-spheres.

Σ^3 and Smooth Structures

Theorem (F. Gonzalez-Acuña'70)

If $n \neq 3$ then

$$\Theta^n \cong \Theta^n_{\mathbb{Z}},$$

where Θ^n is the cobordism group of *homotopy* n-spheres, $\Theta^n_{\mathbb{Z}}$ is the one of *homology* n-spheres.

In particular, the groups $\Theta^n_{\mathbb{Z}}$ are finite for all $n \neq 3$.

Theorem (K. Hendricks et al.'21)

Let Θ^3_{SF} be the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by the homology Seifert spheres. Then the quotient group $\Theta^3_{\mathbb{Z}}/\Theta^3_{\mathrm{SF}}$ has $\mathbb{Z}^\infty$ as a subgroup.

Kervaire and Novikov's Result for Σ^4

Theorem (M. Kervaire'69, S. Novikov'62 (unpubl.))

Every balanced superperfect finitely presented group can be the fundamental group of a homology 4-sphere.

E.g., $SL(2, \mathbb{F}_p)$ for $p > 3$ (the author thanks Peter Teichner for this observation) and Hig_n for $n \geqslant 4$ serve as fundamental groups of homology 4-spheres.

Corollary

The fundamental group of any Σ^3 can serve as the fundamental group of some Σ^4.

Another way to get the corollary: the Suciu *1-spin* $\sigma_1\left(\Sigma^3\right)$.

Deficiency of $\pi_1(\Sigma^4)$

Question

Is every acyclic group the fundamental group of some homology 4-sphere?

It seems that the universal acyclic finitely presented group would serve as a counterexample.

Theorem (J. Hillman'02)

There is a homology 4-sphere with the fundamental group of deficiency -1.

Theorem (C. Livingston'03)

For any $N > 0$, there exists a homology 4-sphere whose fundamental group of deficiency smaller than $-N$.

Finite $\pi_1(\Sigma^4)$

Suppose the homology 4-sphere Σ^4 has finite nontrivial fundamental group.

According to Donaldson's theory $\widetilde{\Sigma^4}$ is homeomorphic to

$$(\mathbb{C}P^2)^{\#m},\ (\overline{\mathbb{C}P^2})^{\#n},\ \text{or}\ (\pm\mathcal{M}_{E_8})^{\#p}\#(\mathbb{S}^2\times\mathbb{S}^2)^{\#q},$$

where # — the connected sum of manifolds, $\mathcal{M}_{E_8}$ — Milnor's E_8-manifold.

For the signatures we have $\sigma(\widetilde{\Sigma^4}) = |\pi_1(\Sigma)|\cdot\sigma(\Sigma^4) = 0$. Hence,

Proposition (K.)

For Σ^4 with finite nontrivial $\pi_1(\Sigma^4)$, we have

$$\widetilde{\Sigma^4} \cong \left(\mathbb{S}^2\times\mathbb{S}^2\right)^{|\pi_1(\Sigma^4)|-1}.$$

Finite $\pi_1(\Sigma^4)$

Question

How many homology spheres are there with a fixed finite fundamental group?

Theorem (I. Hambleton, M. Kreck'88)

Let π and χ be a finite group and integer, respectively. Then, there is only a *finite number* of closed orientable 4-manifolds with the fundamental group π and the Euler characteristic χ up to homeomorphism.

Corollary

Let π be a finite group. Then, there are *only finitely many* homology 4-spheres with the fundamental group π up to homeomorphism.

Homology Spheres With Fixed π_1

As follows from the results of A. Suciu'90, the corollary above doesn't hold for $|\pi| = \infty$.

For M^n and $p > 0$ we get the *p-spin of M*:

$$\sigma_p M^n := \partial\left(M_0 \times \mathbb{D}^{p+1}\right) = M_0 \times \mathbb{S}^p \bigsqcup_{\mathbb{S}^{n-1} \times \mathbb{S}^p} \mathbb{S}^{n-1} \times \mathbb{D}^{p+1},$$

where $M_0 = M \setminus \operatorname{Int} \mathbb{D}^n$.

Theorem (A. Suciu'90)

For $n \geqslant 3$ and $N \geqslant 2$ there are N homology n-spheres with isomorphic π_1's and π_2's as $\mathbb{Z}\pi_1$-modules, but with different k-invariants.

Groups That Are Not $\pi_1(\Sigma^4)$

Theorem (J.-C. Hausmann, Sh. Weinberger'85)

There are nontrivial superperfect finitely presented groups, both with and without torsion, which cannot serve as fundamental groups of homology 4-spheres.

The case of a non-torsion group uses the *Kan–Thurston construction*.

Kan-Thurston Construction

Theorem (D. Kan, W. Thurston'76)

For every path connected $X \in s\mathcal{S}et_*$, *there exists*

$$t : K(G_X, 1) \simeq TX \to X,$$

such that

$$H_*(TX; t^*\mathcal{A}) \cong H_*(X; \mathcal{A}), \; H^*(TX; t^*\mathcal{A}) \cong H^*(X; \mathcal{A})$$

for every local coefficient system $\mathcal{A}$ *on* X.

It is based on the *ad hoc* constructions of acyclic group cones.

Categorical Meaning

Corollary (A. Deleanu'82)

$$\pi_0 \mathcal{CW} \cong \mathcal{GP}[\Gamma^{-1}].$$

Objects of $\mathcal{GP}$ are pairs (G, P), $P \lhd G$, $H_1(P) = 0$.
Morphisms of $\mathcal{GP}$ are homomorphisms of pairs $f : (G, P) \to (G', P')$ for which $f(P) \subset P'$.

The set of morphisms Γ consists of those morphisms $f : (G, P) \to (G', P')$ such that $f : G/P \cong G'/P'$ and $f_* : H_*(G; \mathcal{A}) \cong H_*(G'; \mathcal{A})$ for any G'/P'-module $\mathcal{A}$.

Functors of the Quillen plus and Kan–Thurston construction are *inverse* in some sense.

∞-topos Meaning?

Theorem (M. Hoyois'19)

$\mathcal{X}$ — ∞-topos where π_1 preserves products. Then, there is an adjunction

$$\mathcal{X} \underset{\mathrm{id}\times\{1\}}{\overset{(-)^+}{\underset{\perp}{\leftrightarrows}}} \mathcal{X}^{\lozenge},$$

$\mathcal{X}^{\lozenge}$ — an ∞-category of pairs (X, P), $X \in \mathcal{X}$ and P is a perfect normal subgroup of $\pi_1(X)$.

Moreover, the map $\mathrm{pr}_1 \circ \eta : X \to (X, P)^+$ is acyclic, η — the unit.

Question

Is there an ∞-topos meaning of the Kan-Thurston construction?

Recognition of Spheres

Question

Is there an algorithm for recognition of the standard sphere $\mathbb{S}^n$?

Theorem (H. Rubinstein, A. Thompson'94)

Yes, there is for $n = 3$.

Theorem (S. Novikov'62)

The property of an n-dimensional manifold to be a standard n-dimensional sphere ($n \geqslant 5$) or the property of a contractible region in an $(n+1)$-dimensional Euclidean space with a smooth boundary to be the ordinary $(n+1)$-disk, *unrecognizable*.

Question

What about homology 4-spheres?

Thank you!