On the (non)-multiplicativity of the signature

Let $M^{2n} \to E \xrightarrow{\pi} B^{2d}$ bundle ...

Question: $\mathrm{sd}(\pi) = \mathrm{sign}(E) - \mathrm{sign}(M) \cdot \mathrm{sign}(B)$ what is this?

Flavours:

- M mfd $\quad B^{2d} \xrightarrow{\pi} B\mathrm{Diff}^+(M)$ or $B\mathrm{Homeo}^+(M)$
- $M = X$ PD $\quad B^{2d} \xrightarrow{\pi} B\mathrm{aut}^+(M)$
- $M = (P, \lambda)$ unimod. form $\quad B^{2d} \xrightarrow{\pi} B\mathrm{Isom}(P, \lambda)$

A) $\mathrm{sd}(\pi)$ is not always zero. $\qquad \mathcal{L} = (H^n(M;\mathbb{R}), \mu_{\mathbb{R}})$

B) when is $\mathrm{sd}(\pi) = 0$? [CHS '57] $\mathrm{sign}(E) = \mathrm{sign}(B;\mathcal{L})$

$\Rightarrow \mathrm{sd}(\pi) = 0$ if $B \to B\mathrm{Isom}(H^n(M;\mathbb{R}), \mu_{\mathbb{R}})$ is trivial.

[Meyer '72] $\quad$ B smooth mfd $\quad B\pi \quad$ π finite group. $(H_*(\pi;\mathbb{Q}) = \mathbb{Q}.)$

c) when is $\mathrm{sd}^{top}(\pi) \equiv 0 \mod k$. eg. $k = 2^\ell$

Ex: · $\mathrm{sign}(E) \equiv \chi(E) \mod 2 \Rightarrow \mathrm{sd}(\pi) \equiv 0 \ (2)$

· $\mathrm{sd}(\pi) \equiv 0 \ (4)$ $\quad \pi : B \to B\mathrm{Aut}^+(X)$ (RW, HKR, K, ...)

$\exists \ \Sigma_h \to E \to \Sigma_g$ w/ $\mathrm{sign}(E) = 4$.

· Conj (Klaus-Teichner, Land) $\mathrm{sd}(\pi) \equiv 0 \mod 4 \cdot 2^k$ if monodromy of $\pi_1 B$ on $H^n(M;\mathbb{Z})/_{\mathrm{tors}} \otimes \mathbb{Z}/_{2^k}$ is trivial

rest of talk: B^{2d} stably framed smooth mfld.

approach: $B \xrightarrow{\pi} B\mathrm{Isom}(P,\lambda) \rightsquigarrow \mathrm{sign}(B;\pi)$

(*) $\Omega^{fr}_{2d}(B\mathrm{Isom}(P,\lambda)) \to \Omega^{fr}_{2d}\Omega^{\infty}\mathrm{GW}^{gs}(\mathbb{Z}) \twoheadrightarrow \boxed{\pi_{2d}\mathrm{GW}^{gs}(\mathbb{Z}) \xrightarrow{\sigma} \pi_{2d}L^{gs}(\mathbb{Z}) \cong \mathbb{Z}}$ $\quad \sigma_{\mathbb{Z}}$

$[\pi] \in \Omega^{fr}_{2d}(B\mathrm{Isom}(P,\lambda)) \longleftrightarrow \mathrm{sign}(B;\pi) \in \pi_{2d}L^{gs}(\mathbb{Z})$

<u>Thm</u> Let $d>0$. Then the image of $\sigma_{\mathbb{Z}}: \pi_{2d}\, GW^{ST}(\mathbb{Z}) \longrightarrow \mathbb{Z}$ is

$$\begin{cases} 2^{d+1}\mathbb{Z} & d \equiv 0,1,3 \quad (4) \\ 2^{d+2}\mathbb{Z} & d \equiv 2 \quad (4) \end{cases}$$

$\exists$ a variant for odd base dimension.

geometric analog of $(*)$ is

$(*)$ $\Omega^{fr}_{2d}(BDiff^+(M)) \to \Omega^{fr}_{2d}(\Omega^\infty MTSO(2n)) \to \pi_{2d} MTSO(2n) \to \pi_{2d+2n} MSO \to \mathbb{Z}$

Fact: $\pi_{4k-2} MTSO(2) \to \mathbb{Z}$ is p-adically surj. $\Leftrightarrow$ $(p, 2k)$ is irreg pair