

On the (non)-multiplicativity of the signature

Let $M^{2n} \rightarrow E \xrightarrow{\pi} B^{2d}$ bundle ..

Question:

$$\boxed{\text{sd}(\pi) = \text{sign}(E) - \text{sign}(M) \cdot \text{sign}(B)} \quad \text{what is this?}$$

flavours:

- M mfd $B^{2d} \xrightarrow{\pi} \text{BDiff}^+(M)$ or $\text{BHome}^+(M)$
- $M = X$ PD $B^{2d} \xrightarrow{\pi} \text{Baut}^+(M)$ ↙
- $M = (P, \mathcal{A})$ upholstered form $B^{2d} \xrightarrow{\pi} \text{BIsom}(P, \mathcal{A})$ ↓

A) $\text{sd}(\pi)$ is not always zero.

$$\mathcal{L} = (\mathbb{H}^n(M; \mathbb{R}), \mu_{\mathbb{R}})$$

B) when is $\text{sd}(\pi) = 0$? [ChS '57] $\text{sign}(E) = \text{sign}(B; \mathcal{L})$

$\Rightarrow \text{sd}(\pi) = 0$ if $B \rightarrow \text{BIsom}(\mathbb{H}^n(M; \mathbb{R}), \mu_{\mathbb{R}})$ is trivial.
[Meyer '72] B moduli mfd $\rightarrow \text{B}\pi \rightarrow \pi$ finite group. ($H_*(\pi; \mathbb{Q}) = \mathbb{Q} \cdot$)

c) when is $sd(\pi) \equiv 0 \pmod{k}$. eg. $k=2^l$

Ex: \cdot $sign(E) \equiv \chi(E) \pmod{2} \Rightarrow sd(\pi) \equiv 0 \pmod{2}$

\cdot $sd(\pi) \equiv 0 \pmod{4}$ $\pi: B \rightarrow B_{out}^+(X)$ (RW, HKR, K, ...)

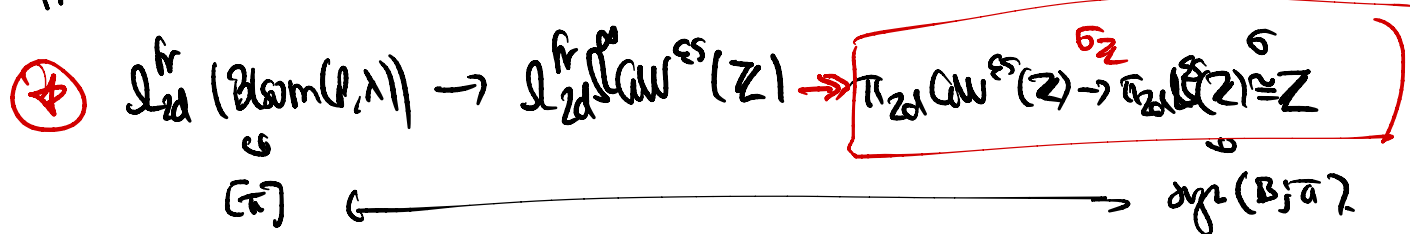
$\exists \Sigma_u \rightarrow E \rightarrow \Sigma_g$ w/ $sign(E) = 4$.

Conj (Klaus-Teichner, Land) $sd(\pi) \equiv 0 \pmod{4 \cdot 2^k}$ \nleftrightarrow

monodromy of $\pi_2 B$ on $H^*(M; \mathbb{Z}) / \text{tors} \otimes \mathbb{Z}/2^k$ is trivial

rest of talk: B^{2d} stably framed smooth mfld.

approach: $B \xrightarrow{\pi} B(\text{Spin}(p, \Lambda)) \hookrightarrow sign(B; \pi)$



Then let $d > 0$: then the image of $\sigma_{\mathbb{Z}}: \pi_{2d} \text{Grass}^2(\mathbb{Z}) \rightarrow \mathbb{Z}$ is

$$\left\{ \begin{array}{l} \mathbb{Z}^{d+1} \mathbb{Z} \\ \mathbb{Z}^{d+2} \mathbb{Z} \end{array} \right. \quad \begin{array}{l} d \equiv 0, 3 \quad (4) \\ d \equiv 2 \quad (4) \end{array}$$

For a variant for odd base dimension.

geometric analog of is

$$(*) \quad \Omega_{2d}^{\text{fr}}(\text{BSpin}^+(4n)) \rightarrow \Omega_{2d}^{\text{fr}}(\mathbb{R}^{\infty} \text{MISO}(2n)) \rightarrow \pi_{2d} \text{MISO}(2n) \rightarrow \pi_{2d+2n} \text{MSO} \rightarrow \mathbb{Z}$$

Fact: $\pi_{4k-2} \text{MISO}(2) \rightarrow \mathbb{Z}$ is p -adically div. $\Leftrightarrow (p, 2k)$ is
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