Norm, Assembly and Coassembly

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Malkiewich's Theorem

Let G be a finite group, and R a ring (spectrum). We want to understand the map

asbly :
$$BG_+ \wedge K(R) \rightarrow K(R[G])$$
.

Malkiewich constructs further maps

$$\mathcal{K}(R[G]) \xrightarrow{\mathsf{Cartan}} \mathcal{G}(R[G]) \xrightarrow{\mathsf{coasbly}} \mathsf{map}(BG, \mathcal{K}(R)),$$

and shows

Theorem

The composition

 $BG_{+} \land K(R) \xrightarrow{asbly} K(R[G]) \xrightarrow{Cartan} \mathcal{G}(R[G]) \xrightarrow{coasbly} map(BG, K(R))$

agrees with the norm map for K(R) equipped with the trivial *G*-action.

$$BG_+ \wedge K(R) \xrightarrow{\text{asbly}} K(R[G]) \xrightarrow{\text{Cartan}} \mathcal{G}(R[G]) \xrightarrow{\text{coasbly}} \operatorname{map}(BG, K(R))$$

agrees with the norm map for K(R) equipped with the trivial *G*-action. Norm for <u>finite</u> groups is equivalence on rational or K(n)-local spectra:

Corollary

The map asbly : $BG_+ \wedge K(R) \rightarrow K(R[G])$ is rationally and K(n)-locally injective.

Compare:

Conjecture (Novikov)

The map asbly : $BG_+ \wedge L(R) \rightarrow L(R[G])$ is rationally injective for all groups G.

What is the assembly map?

• K-theory is a (universal) functor

$$K: \mathsf{Cat}^{perf} \to \mathsf{Sp}.$$

- $K(R) = K(\operatorname{Perf}_R)$
- Fact: Consider $Perf_R$ with trivial G-action. Then

$$(\operatorname{Perf}_R)_{hG} = \operatorname{Perf}_{R[G]}.$$

(Homotopy orbits!)

• Fact: Assembly is nothing more than the natural comparison map

$$K(\operatorname{Perf}_R)_{hG} \to K((\operatorname{Perf}_R)_{hG})$$

• Remark: This holds for all G, not necessarily finite.

• Fact: Consider $Perf_R$ with trivial G-action. Then

$$(\operatorname{Perf}_R)^{hG} = \operatorname{Fun}(BG, \operatorname{Perf}_R)$$

(Homotopy fixed points!)

• Fact: Co-assembly is nothing more than the natural comparison map

$$K((\operatorname{Perf}_R)^{hG}) \to K(\operatorname{Perf}_R)^{hG}$$

Definition

Let $X \in C$ be an object with (naive) *G*-action, and

 $F:\mathcal{C}\to\mathcal{D}$

be a functor. The assembly map for $F(X_{hG})$ is the canonical map

$$F(X)_{hG} \to F(X_{hG}).$$

(Homotopy orbits)

Definition

Let $X \in C$ be an object with (naive) *G*-action, and

 $F:\mathcal{C}\to\mathcal{D}$

be a functor. The <u>co-assembly</u> map for $F(X^{hG})$ is the canonical map

 $F(X^{hG}) \to F(X)^{hG}$.

(Homotopy fixed points)

 $X \in \mathcal{C}$, object with (naive) *G*-action. A norm map for X is a map

nm :
$$X_{hG} \to X^{hG}$$
.

Equivalently, a (homotopy) fixed point

$$\mathsf{nm} \in \mathsf{Map}(X, X)^{h(G \times G)}$$

Classical example: A an abelian group with G-action

$$nm = \sum_{g \in G} g \cdot : A \to A$$

E_{∞} -monoids and group actions

Ingredients we need:

- "Commutative monoids" to form sums
- Take E_{∞} -monoids.
- Fact: An E_∞ -monoid is equivalently a functor

 $\mathsf{Span}(\mathsf{Fin})^{\textit{op}} \to \mathcal{C}$

preserving finite products.

- "Naive G-actions" to have actions by elements of G
- Fact: Object with naive G-action is equivalently a functor

$$\operatorname{Free}_{G}^{op} \to \mathcal{C}$$

preserving finite products.

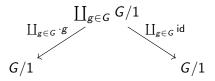
The universal norm map

- Put them into the mixer:
- Fact: An E_{∞} -monoid with naive G-action is equivalently a functor

$$\mathsf{Span}(\mathsf{Free}_G)^{op} o \mathcal{C}$$

preserving finite products.

• Universal norm: The span " $\sum_{g \in \mathcal{G}} \cdot g$ ",



Theorem

Let $X \in C$ be an E_{∞} -monoid with (naive) G-action, and

 $F:\mathcal{C}\to\mathcal{D}$

be a functor preserving finite products. Then there exists a commutative square

Apply the previous theorem to many contexts:

- Any additive invariant $Cat^{perf} \rightarrow Sp$, such as K, THH, TC, etc.
- Any additive invariant $\operatorname{Cat}^{\operatorname{poinc}} \to \operatorname{Sp}$, such as GW or L-theory.
- We can incorporate *G*-actions on *R* (twisted group rings)

Corollary

In all of the above situations, Malkiewich's theorem generalizes. In particular, assembly is rationally and K(n)-locally injective for all of the above.

Thank you for your attention!