Norm, Assembly and Coassembly

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Malkiewich's Theorem

Let G be a finite group, and R a ring (spectrum). We want to understand the map

$$\text{asbly} : BG_+ \wedge K(R) \to K(R[G]).$$

Malkiewich constructs further maps

$$K(R[G]) \xrightarrow{\text{Cartan}} \mathcal{G}(R[G]) \xrightarrow{\text{coasbly}} \text{map}(BG, K(R)),$$

and shows

Theorem

The composition

$$BG_+ \wedge K(R) \xrightarrow{asbly} K(R[G]) \xrightarrow{Cartan} \mathcal{G}(R[G]) \xrightarrow{coasbly} map(BG, K(R))$$

agrees with the norm map for $K(R)$ equipped with the trivial G-action.

Immediate Corollary

$$BG_+ \wedge K(R) \xrightarrow{\text{asbly}} K(R[G]) \xrightarrow{\text{Cartan}} \mathcal{G}(R[G]) \xrightarrow{\text{coasbly}} \mathrm{map}(BG, K(R))$$

agrees with the norm map for $K(R)$ equipped with the trivial G-action. Norm for finite groups is equivalence on rational or $K(n)$-local spectra:

Corollary

The map $\mathrm{asbly}: BG_+ \wedge K(R) \to K(R[G])$ *is rationally and* $K(n)$*-locally injective.*

Compare:

Conjecture (Novikov)

The map $\mathrm{asbly}: BG_+ \wedge L(R) \to L(R[G])$ *is rationally injective for all groups* G*.*

What is the assembly map?

- K-theory is a (universal) functor

$$K : \mathrm{Cat}^{perf} \to \mathrm{Sp}.$$

- $K(R) = K(\mathrm{Perf}_R)$
- Fact: Consider Perf_R with trivial G-action. Then

$$(\mathrm{Perf}_R)_{hG} = \mathrm{Perf}_{R[G]}.$$

  (Homotopy orbits!)
- Fact: Assembly is nothing more than the natural comparison map

$$K(\mathrm{Perf}_R)_{hG} \to K((\mathrm{Perf}_R)_{hG})$$

- Remark: This holds for all G, not necessarily finite.

What is the co-assembly map?

- Fact: Consider Perf_R with trivial G-action. Then

$$(\mathrm{Perf}_R)^{hG} = \mathrm{Fun}(BG, \mathrm{Perf}_R)$$

  (Homotopy fixed points!)

- Fact: Co-assembly is nothing more than the natural comparison map

$$K((\mathrm{Perf}_R)^{hG}) \to K(\mathrm{Perf}_R)^{hG}$$

What is an assembly map, anyway?

Definition

Let $X \in \mathcal{C}$ be an object with (naive) G-action, and

$$F : \mathcal{C} \to \mathcal{D}$$

be a functor. The assembly map for $F(X_{hG})$ is the canonical map

$$F(X)_{hG} \to F(X_{hG}).$$

(Homotopy orbits)

What is a co-assembly map, anyway?

Definition

Let $X \in \mathcal{C}$ be an object with (naive) G-action, and

$$F : \mathcal{C} \to \mathcal{D}$$

be a functor. The co-assembly map for $F(X^{hG})$ is the canonical map

$$F(X^{hG}) \to F(X)^{hG}.$$

(Homotopy fixed points)

What is a norm map?

$X \in \mathcal{C}$, object with (naive) G-action. A norm map for X is a map

$$\mathrm{nm} : X_{hG} \to X^{hG}.$$

Equivalently, a (homotopy) fixed point

$$\mathrm{nm} \in \mathrm{Map}(X, X)^{h(G\times G)}.$$

Classical example: A an abelian group with G-action

$$nm = \sum_{g\in G} g\cdot : A \to A$$

E_∞-monoids and group actions

Ingredients we need:

- "Commutative monoids" to form sums
- Take E_∞-monoids.
- Fact: An E_∞-monoid is equivalently a functor

$$\mathrm{Span}(\mathrm{Fin})^{op} \to \mathcal{C}$$

  preserving finite products.
- "Naive G-actions" to have actions by elements of G
- Fact: Object with naive G-action is equivalently a functor

$${\mathrm{Free}_G}^{op} \to \mathcal{C}$$

  preserving finite products.

The universal norm map

- Put them into the mixer:
- Fact: An E_∞-monoid with naive G-action is equivalently a functor
$$\mathrm{Span}(\mathrm{Free}_G)^{op} \to \mathcal{C}$$
preserving finite products.
- Universal norm: The span "$\sum_{g\in G} \cdot g$",

$$G/1 \xleftarrow{\coprod_{g\in G} \cdot g} \coprod_{g\in G} G/1 \xrightarrow{\coprod_{g\in G} \mathrm{id}} G/1$$

A general norm factorization theorem

Theorem

Let $X \in \mathcal{C}$ be an E_∞-monoid with (naive) G-action, and

$$F : \mathcal{C} \to \mathcal{D}$$

be a functor preserving finite products. Then there exists a commutative square

$$\begin{array}{ccc} F(X_{hG}) & \xrightarrow{F(nm)} & F(X^{hG}) \\ \uparrow{\scriptstyle \text{asbly}} & & \downarrow{\scriptstyle \text{co-asbly}} \\ F(X)_{hG} & \xrightarrow{nm} & F(X)^{hG} \end{array}$$

Corollary

Apply the previous theorem to many contexts:

- Any additive invariant $\mathrm{Cat}^{perf} \to \mathrm{Sp}$, such as K, THH, TC, etc.
- Any additive invariant $\mathrm{Cat}^{poinc} \to \mathrm{Sp}$, such as GW or L-theory.
- We can incorporate G-actions on R (twisted group rings)

Corollary

In all of the above situations, Malkiewich's theorem generalizes. In particular, assembly is rationally and $K(n)$-locally injective for all of the above.

Thank you for your attention!