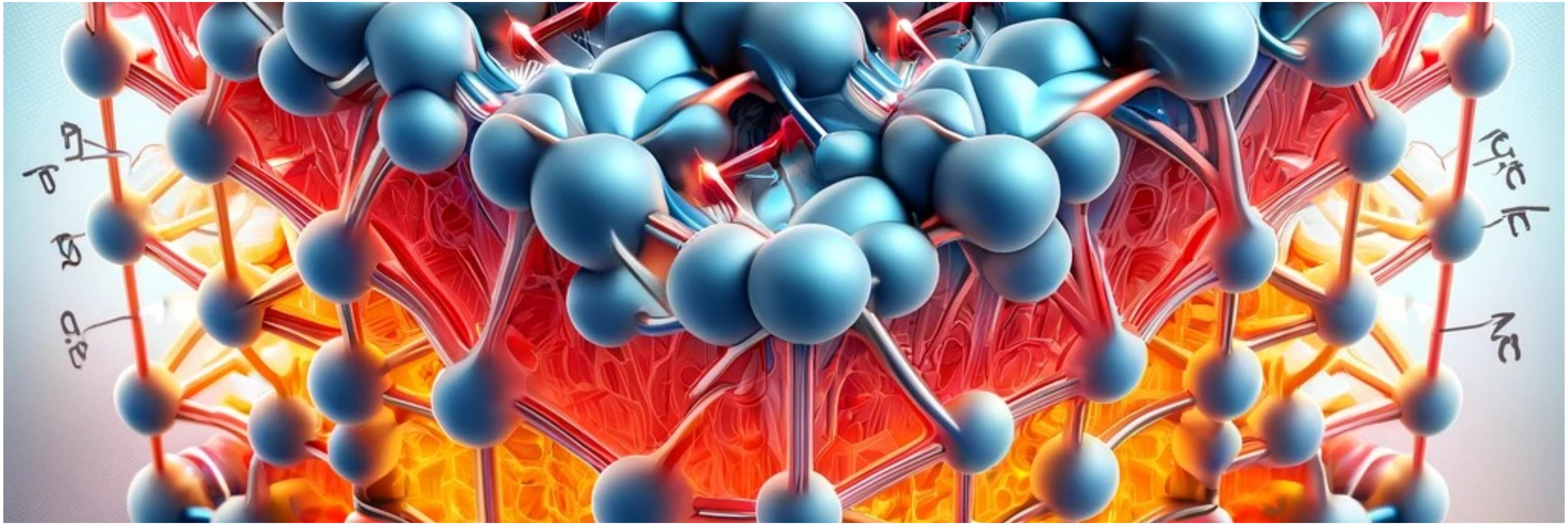


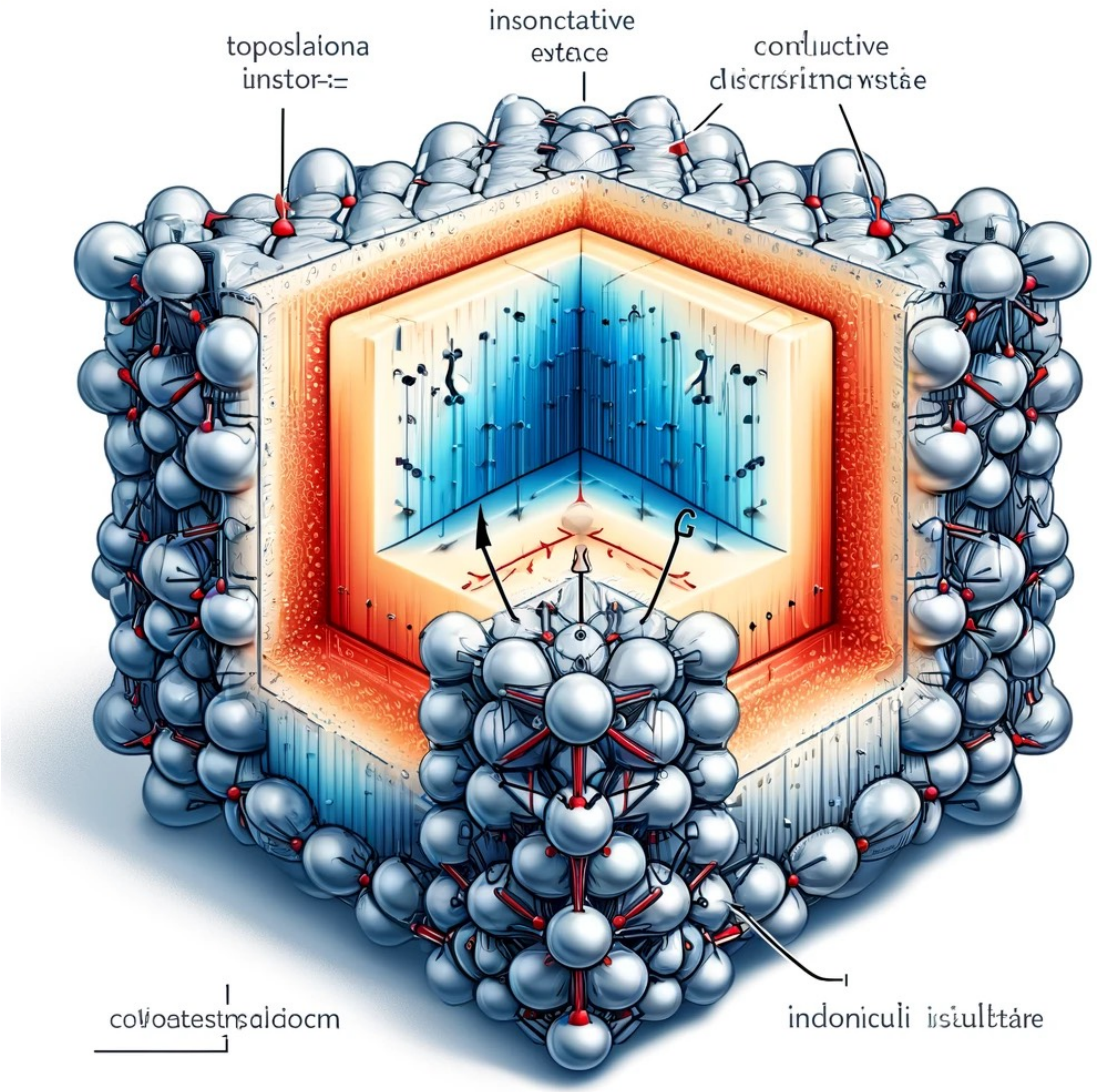
Matthias Ludewig

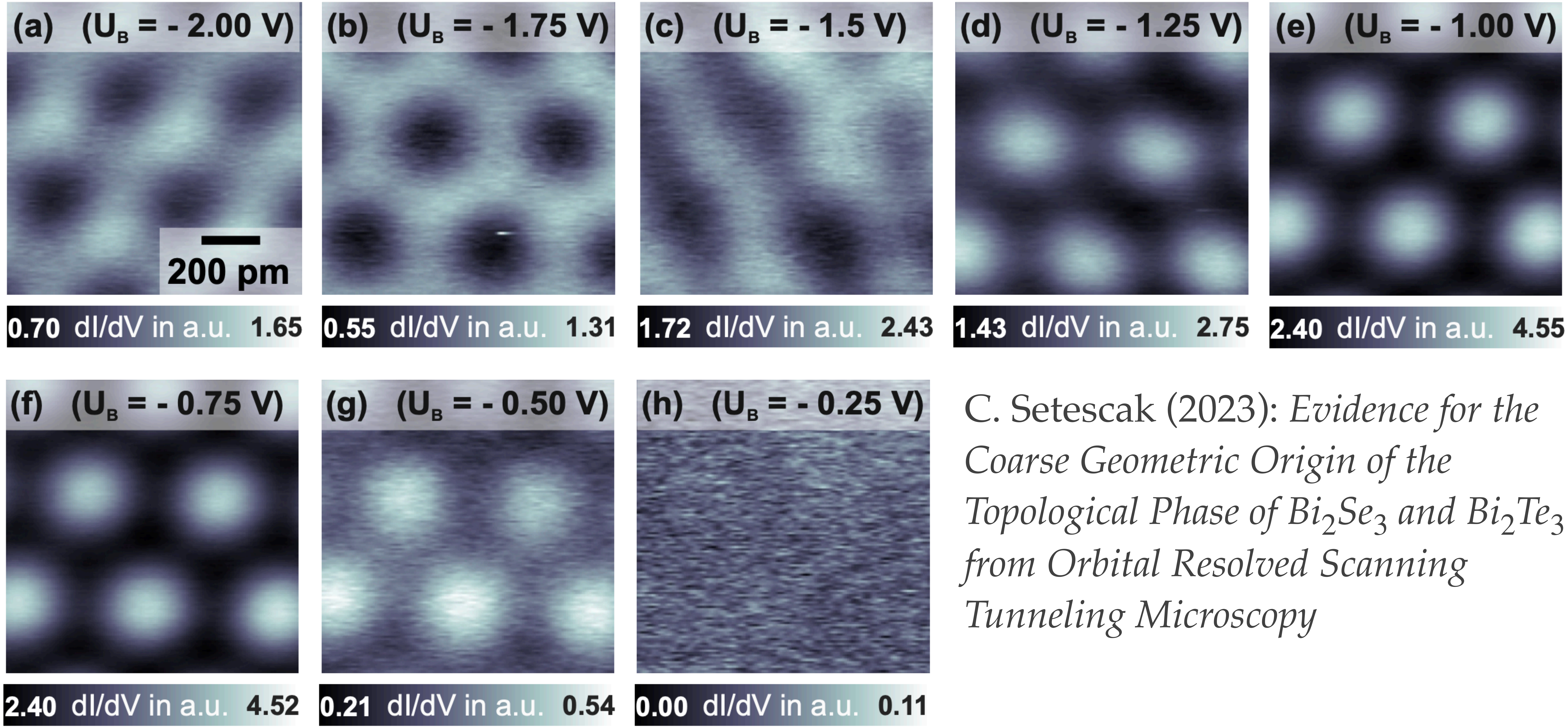
Coarse geometry and topological insulators

Universität Regensburg
SFB 1085 „Higher invariants“

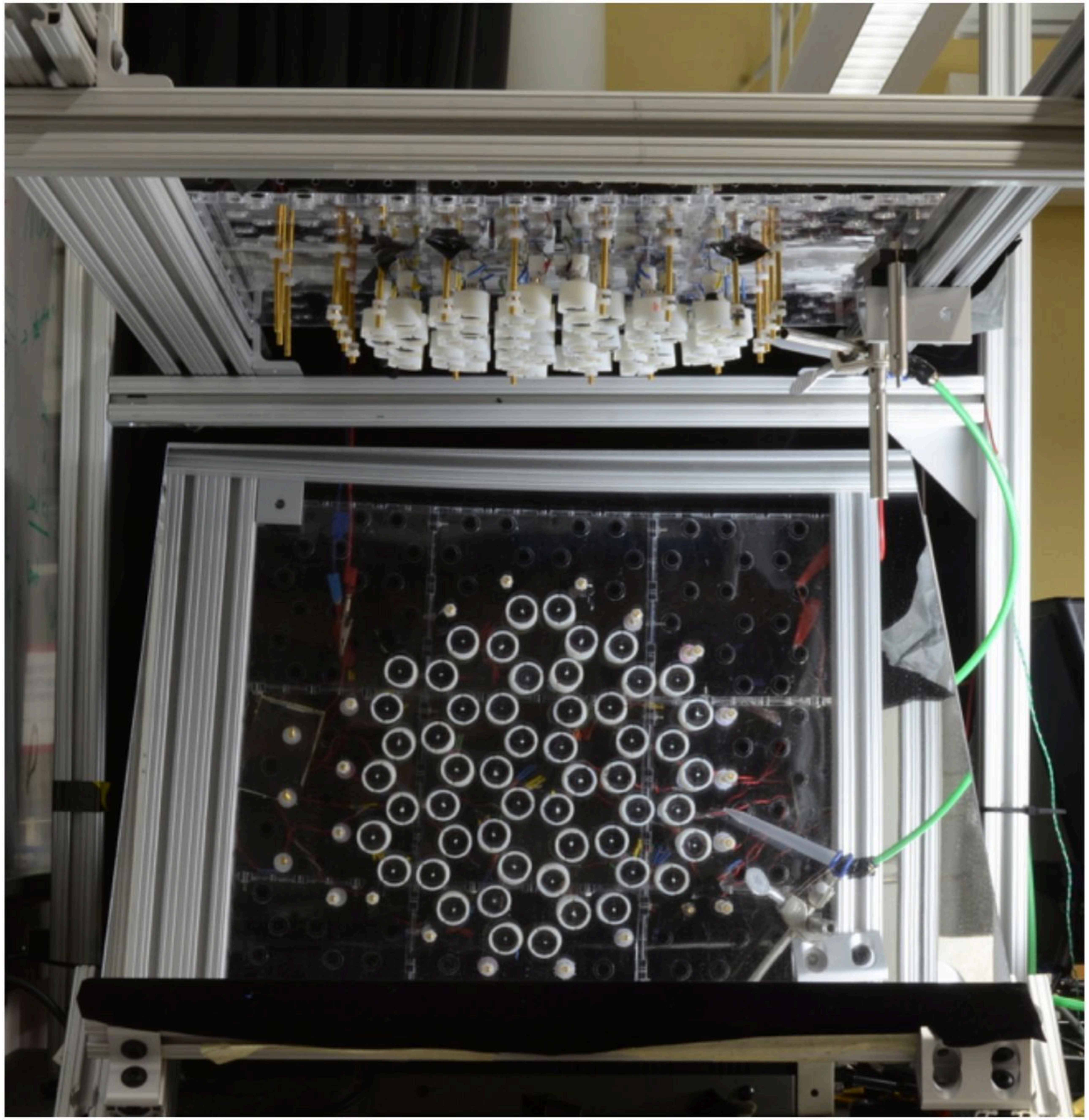
What are topological insulators?

A **topological insulator** is a material whose interior behaves as an electrical insulator while its surface behaves as an electrical conductor,[3] meaning that electrons can only move along the surface of the material.



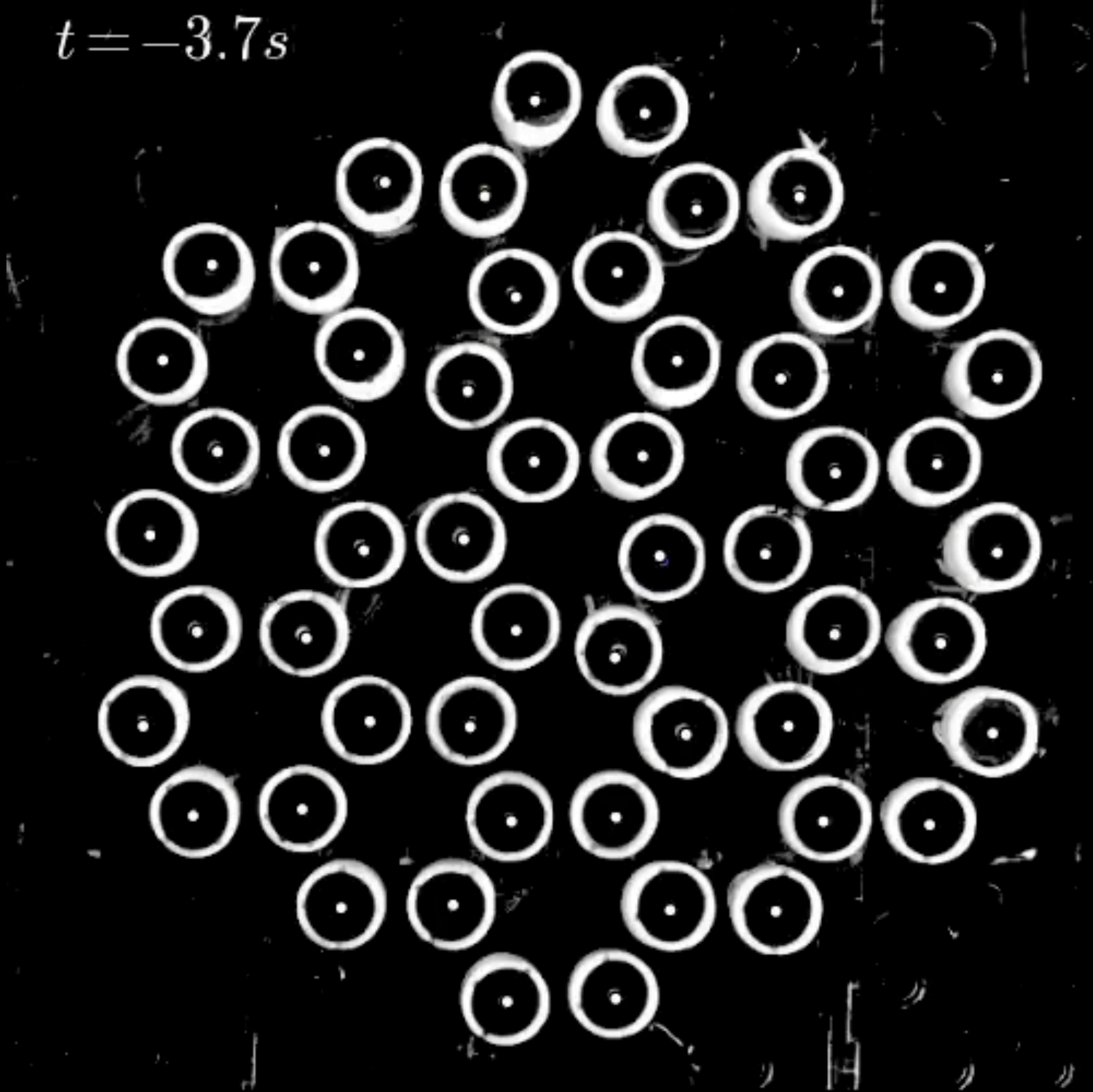


C. Setescak (2023): *Evidence for the Coarse Geometric Origin of the Topological Phase of* Bi_2Se_3 *and* Bi_2Te_3 *from Orbital Resolved Scanning Tunneling Microscopy*

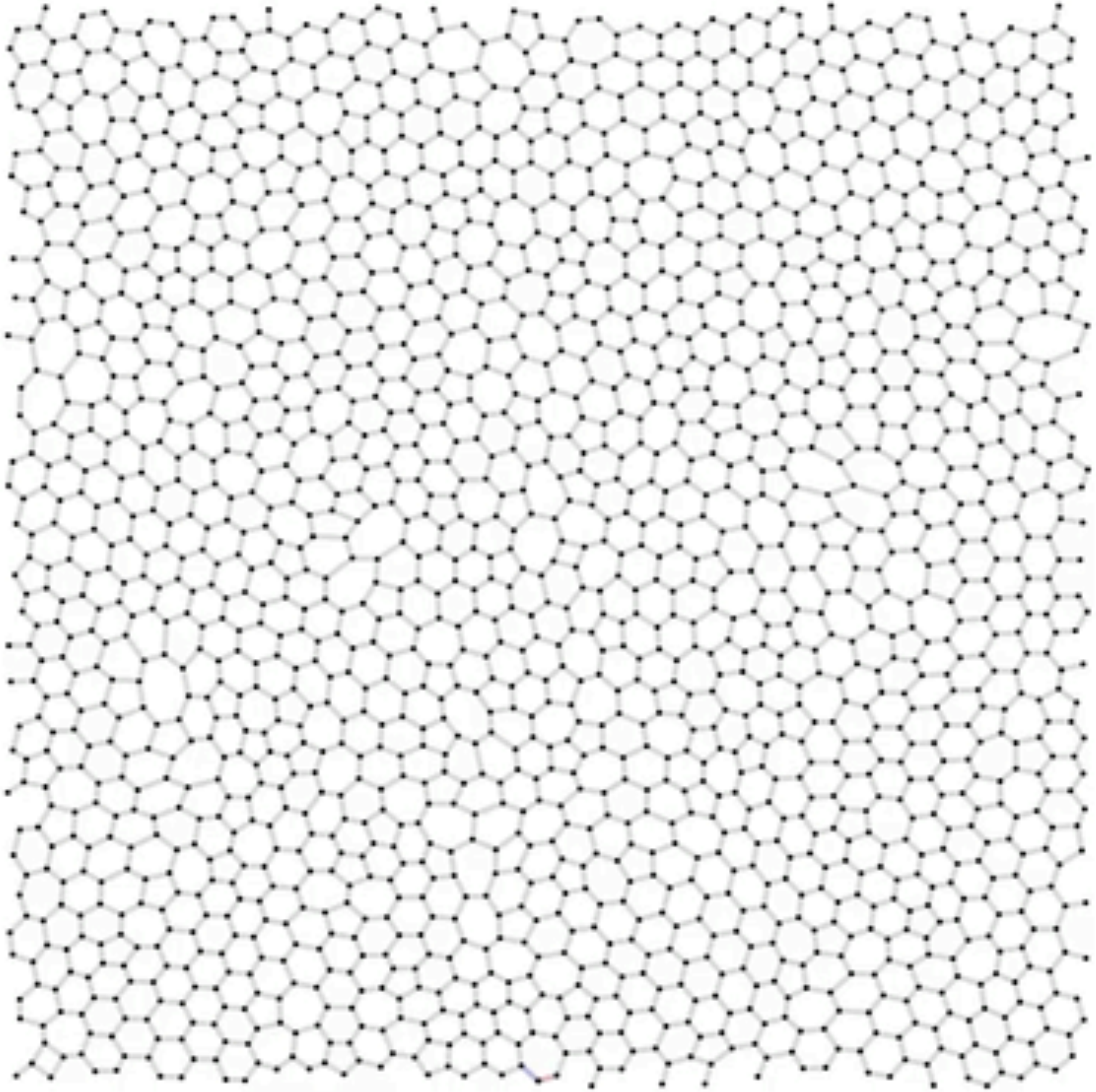


Nash, Kleckner, Read, Vitelli, Turner, Irvine (2015): *Topological mechanics of gyroscopic metamaterials*

Metamaterial constructed from suspended gyroscopes


$t = -3.7s$


$t = -3.0s$



Mitchell, Nash, Hexner, Turner, Irvine (2018): *Amorphous topological insulators constructed from random point sets*

Simulation of metamaterial constructed from random point cloud.

Basic mathematical description

Gapped Hamiltonians

We consider physical systems described by a Hamiltonian H.

self-adjoint (possibly unbounded) operator acting on a Hilbert space $\mathscr{H}$.

Definition. The Hamiltonian H is *insulating* at the energy level E if

$$E \notin \operatorname{spec}(H) .$$

„Fermi energy“

Lattice systems

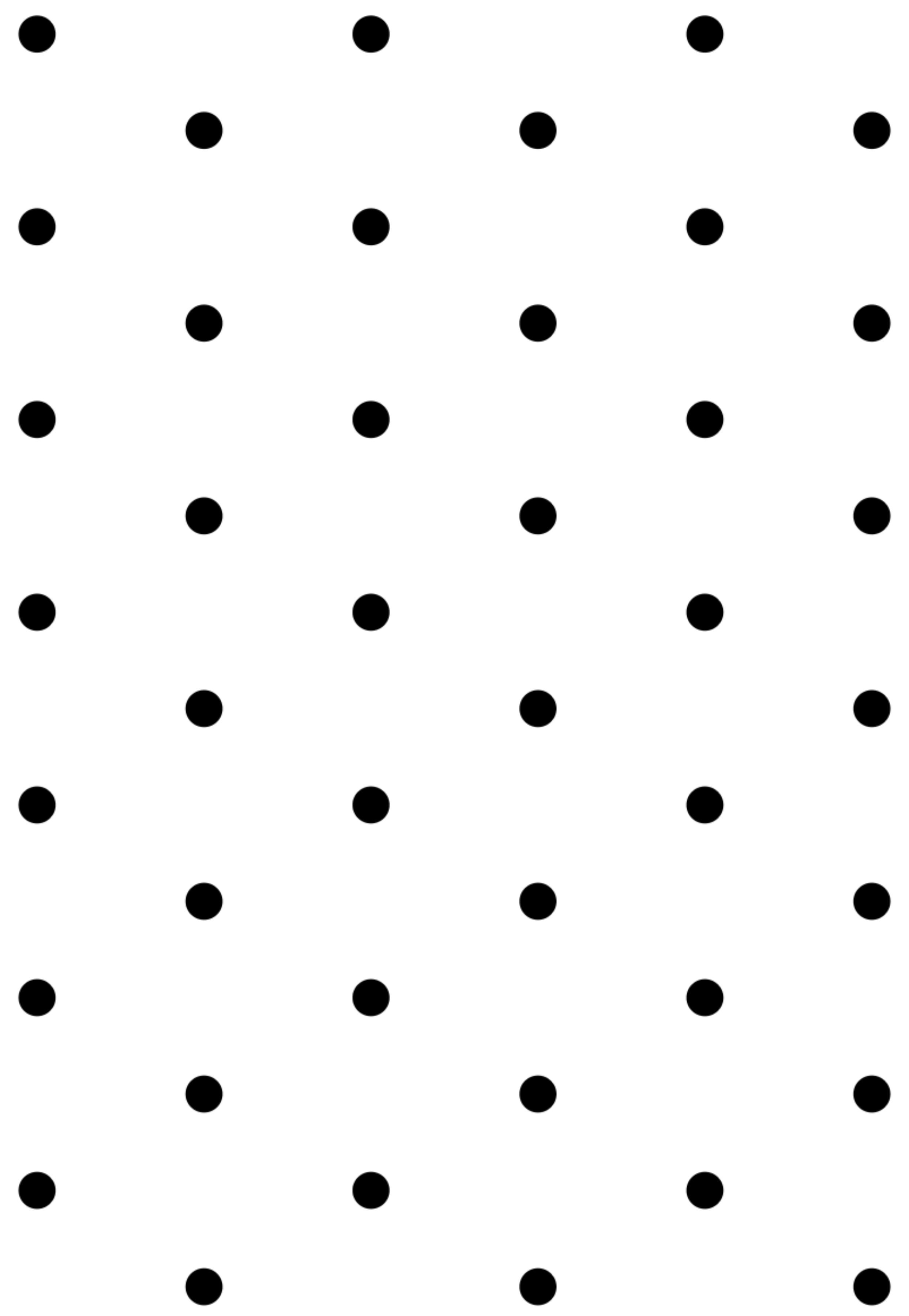
Given a discrete metric space X, one typically considers the Hilbert space

$$\mathscr{H} = \ell^2(X) \otimes \mathbb{C}^n.$$

A typical toy model Hamiltonian for $\Gamma = \mathbb{Z}^d$ would be

$$H = \frac{1}{2i} \sum_{j=1}^{d} (S_j - S_j^*) \otimes \gamma_j + \left(m + \frac{1}{2} \sum_{j=1}^{d} (S_j + S_j^*) \right) \otimes \gamma_0$$

For suitable values of m, this has a spectral gap at $E = 0$.



Continuous systems

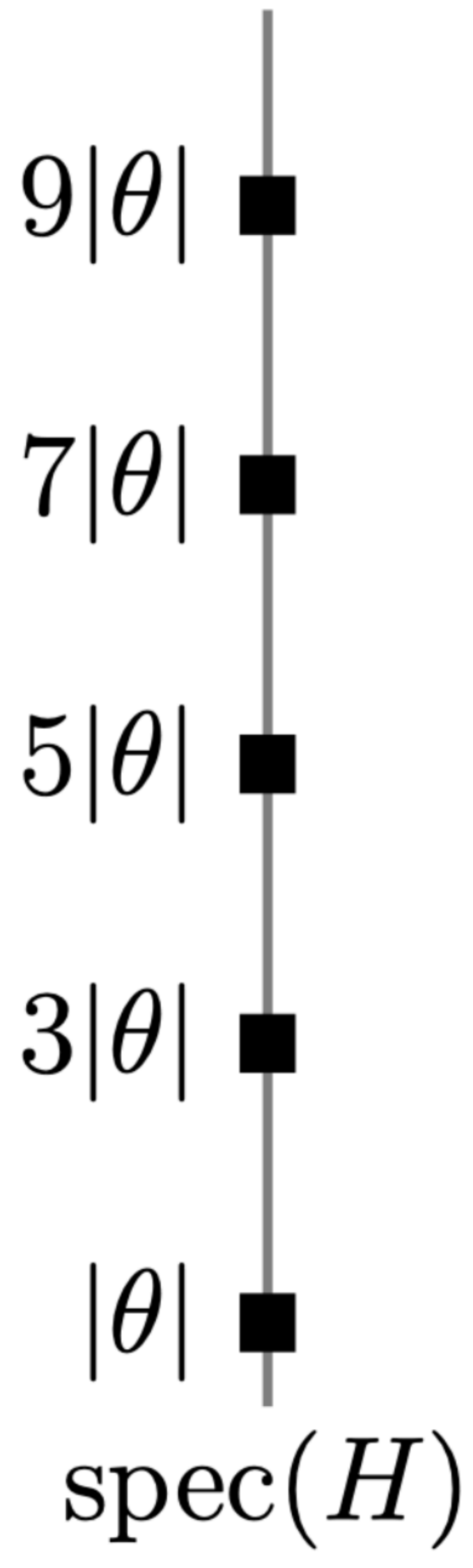
Let X be a Riemannian manifold (typically $X = \mathbb{R}^d$) and consider the Hilbert space $\mathscr{H} = L^2(X)$. The typical Hamiltonians are differential operators (usually Dirac or Laplace type).

A toy example is the **Landau Hamiltonian**,

$$H = (d - iA)^*(d - iA),$$

where $X = \mathbb{R}^2$ and $A \in \Omega^1(\mathbb{R}^2)$ satisfies

$$dA = \theta \cdot \mathrm{vol}, \qquad \theta \in \mathbb{R}.$$



Equivariant case

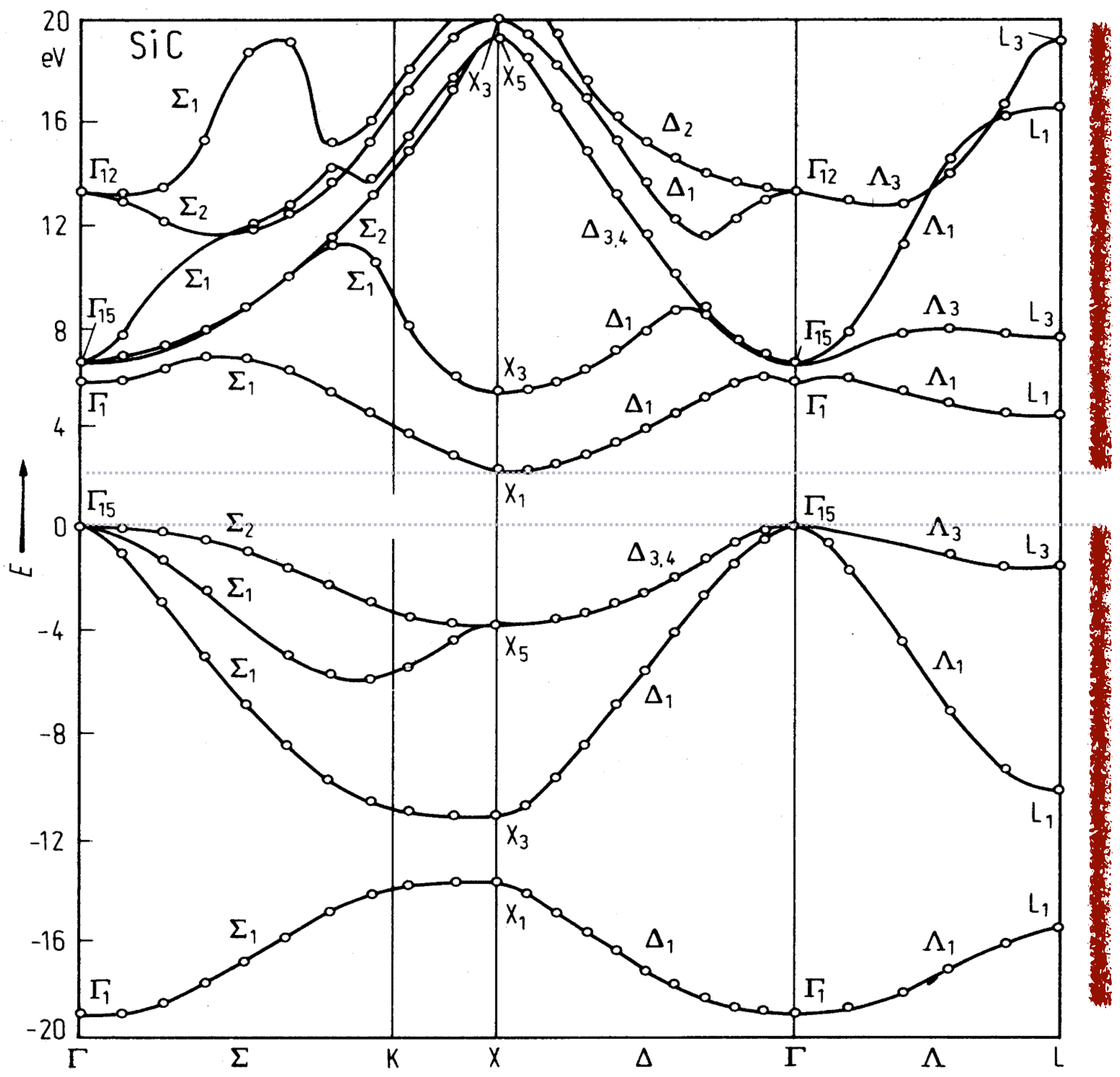
Suppose that $X \subseteq \mathbb{R}^d$ is invariant under translations by $\mathbb{Z}^d$ and that the Hamiltonian H is equivariant.

Fourier transform:

$$\ell^2(X) \otimes \mathbb{C}^d \quad \cong \quad L^2(\mathbb{T}^d) \otimes \mathbb{C}^d$$

$$H \quad \longleftrightarrow \quad \big(H(k)\big)_{k \in \mathbb{T}^d}$$

$$\mathrm{spec}(H) = \bigcup_{k \in \mathbb{T}^d} \mathrm{spec}\big(H(k)\big)$$

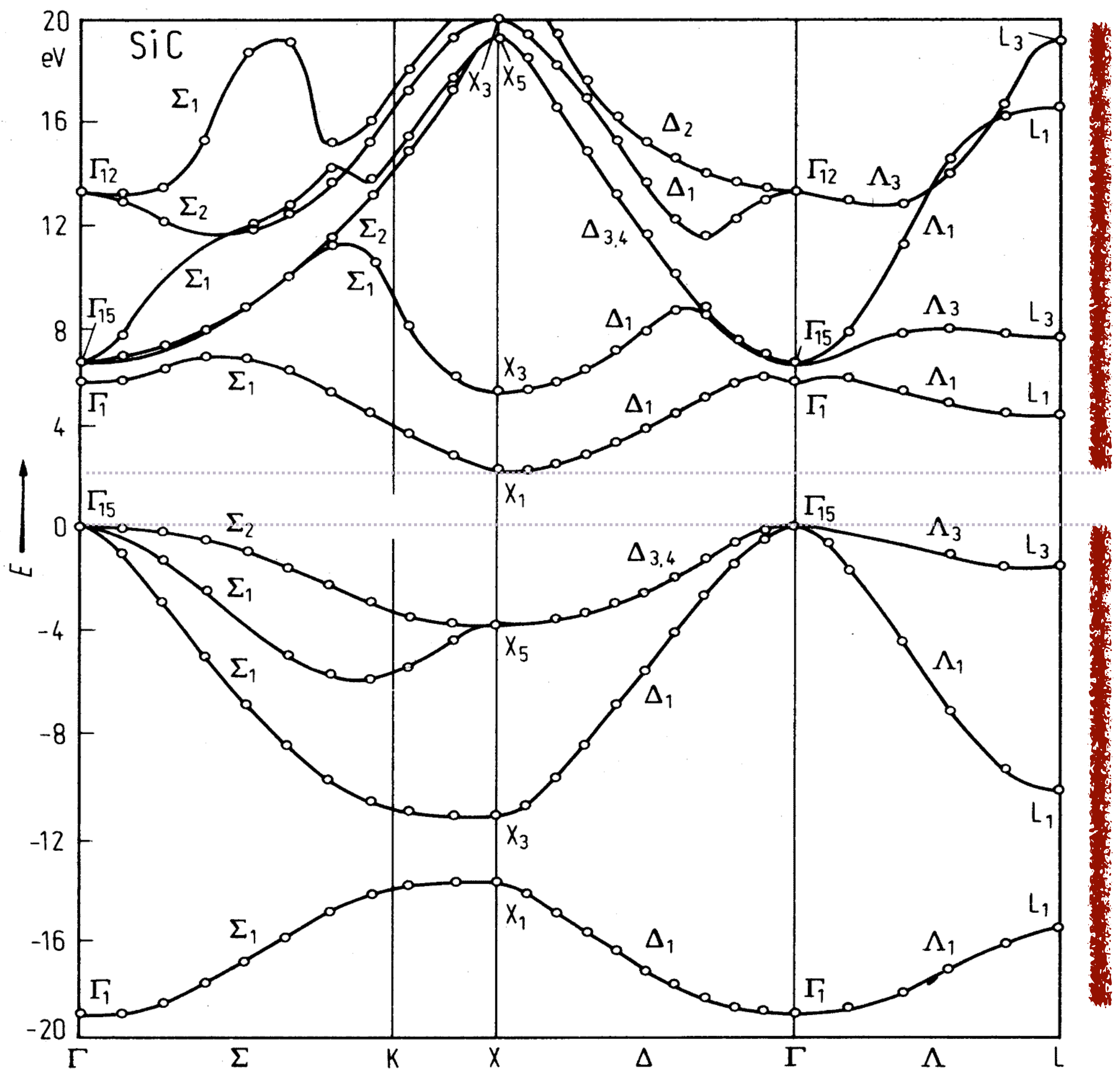


Equivariant case

If H has a spectral gap at E, we may construct a vector bundle V over $\mathbb{T}^d$, with fibers

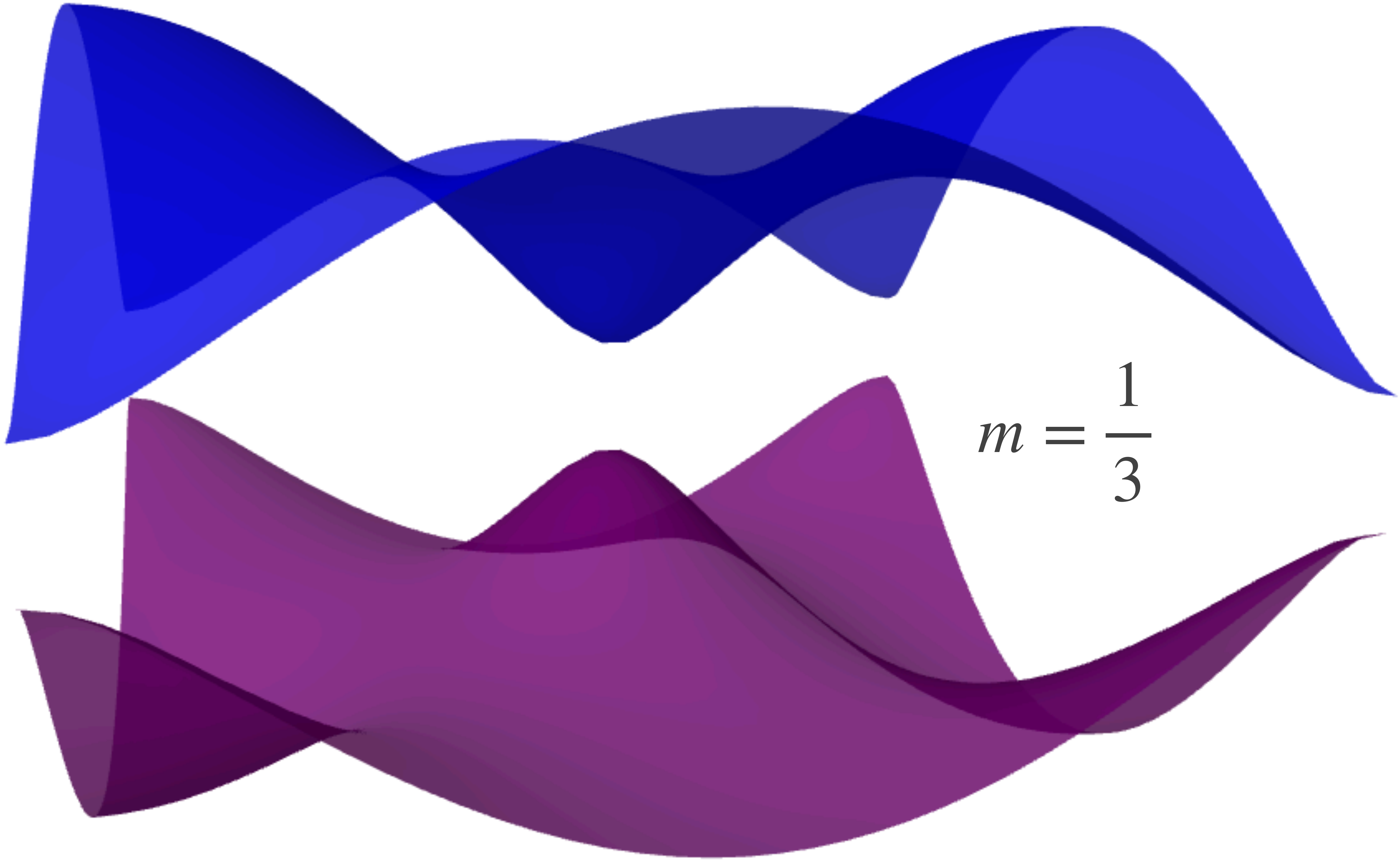
$$V_k = \bigoplus_{\lambda \le E} \mathrm{Eig}(H(k), \lambda).$$

Non-triviality of V may be measured by characteristic classes (*Chern insulator*).



Example: $H(k) = \begin{pmatrix} \cos(k_1) + \cos(k_2) + m & \sin(k_2) - i\sin(k_1) \\ \sin(k_2) + i\sin(k_1) & -\cos(k_1) - \sin(k_2) - m \end{pmatrix}$

Get line bundles with non-trivial first Chern class if $m \in (-2, 0)$ or $m \in (0, 2)$

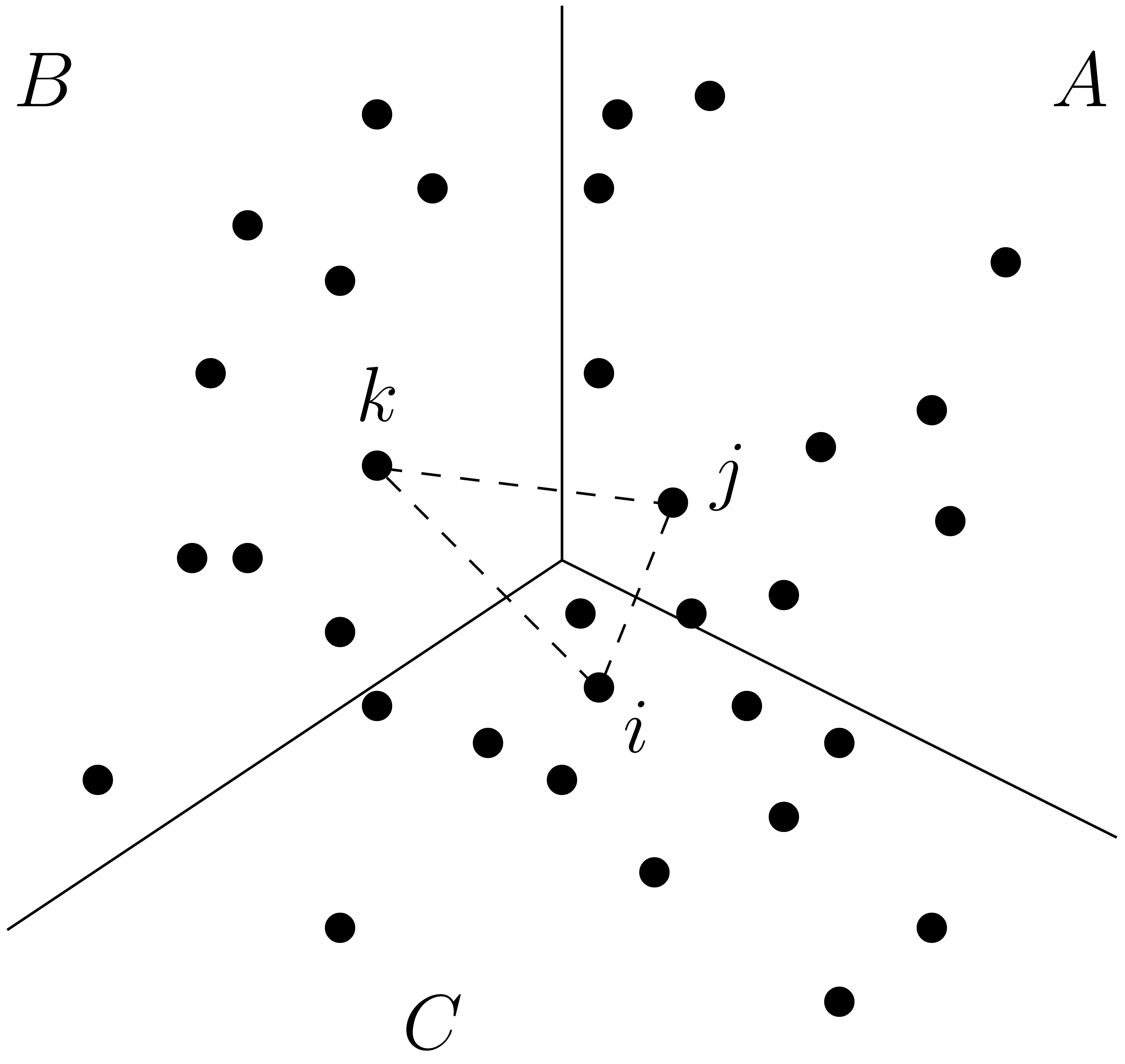


$m = \frac{1}{3}$

Disordered systems

Disordered systems are often considered to be rather mysterious!

There are several *ad hoc* ways to calculate some generalized „Chern number" from the projection P onto the spectrum below the Fermi energy („Fermi projection").



Example. Kitaev's formula

$$\nu(P) = 12\pi i \sum_{x\in A} \sum_{y\in B} \sum_{z\in C} \left(P_{xy}P_{xz}P_{zx} - P_{xz}P_{zy}P_{yx}\right)$$

The non-commutative framework

C^*-algebraic treatment

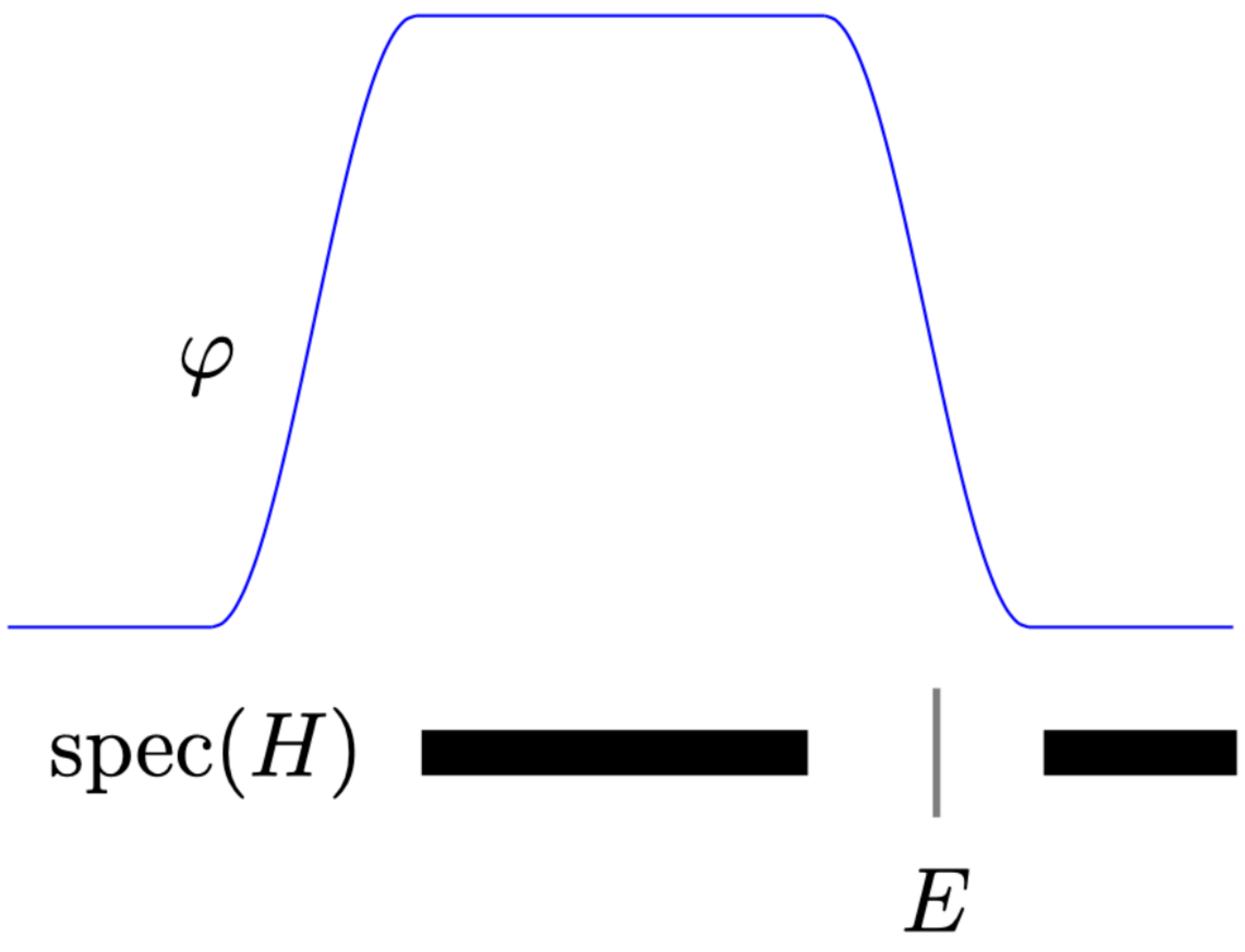
We start with some „observable“ C^*-algebra $\mathscr{A}$ and a Hamiltonian $H \in \mathscr{A}$, which is insulating at E.

or, more generally, a possibly unbounded operator on a Hilbert $\mathscr{A}$-module

The corresponding Fermi projection $P = \varphi(H)$ is an element of $\mathscr{A}$.

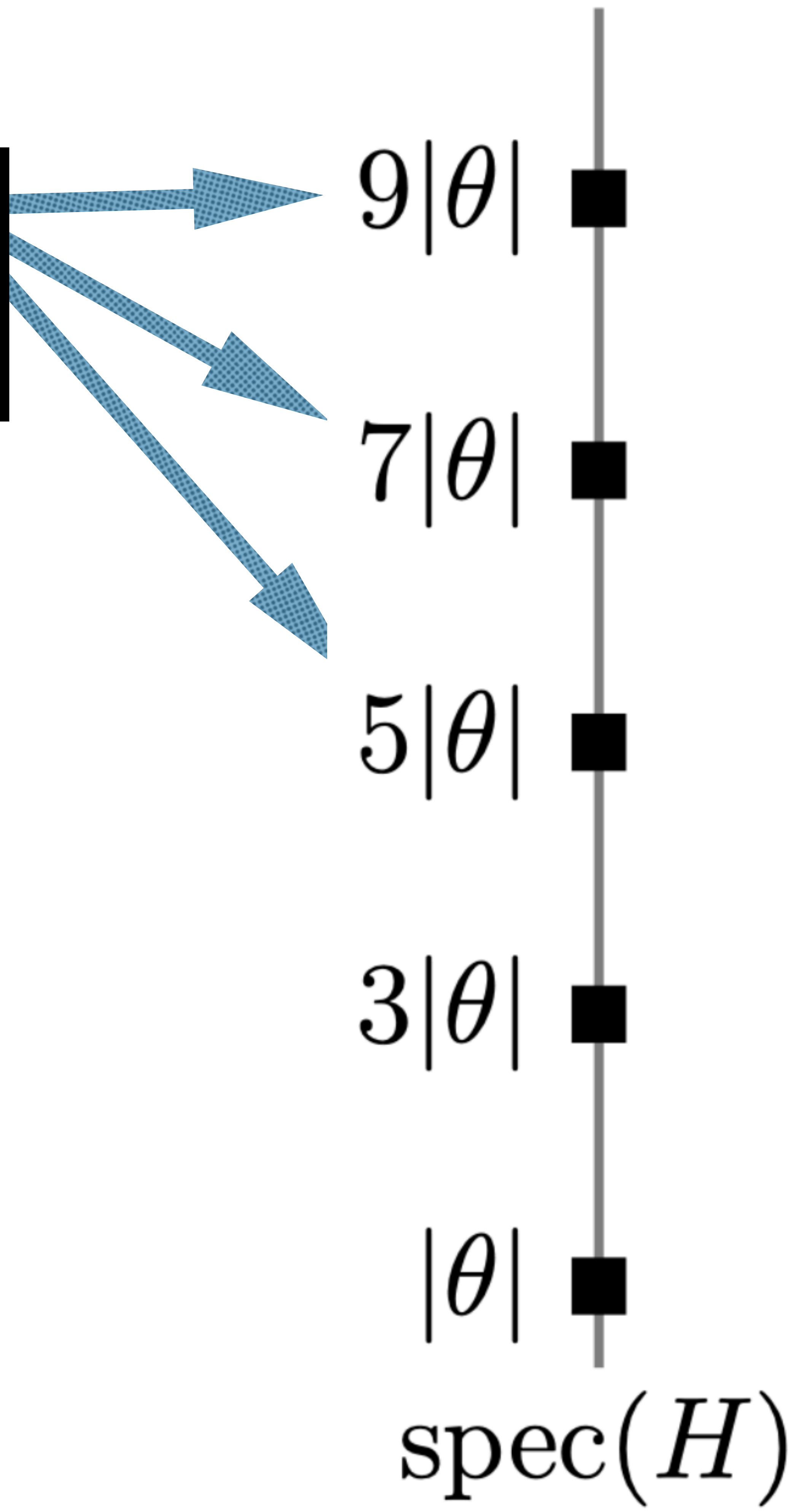
$\Longrightarrow$ Get a class $[P]$ in $K_0(A)$.

Definition. A *topological insulator* is an insulator such that this class $[P]$ is non-trivial.



C^*-algebraic treatment

Example: The spectral projection onto each eigenvalue of the Landau Hamiltonian is a generator of $K_0(C^*(\mathbb{R}^2)) \cong \mathbb{Z}$



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Definition. A *topological insulator* is an insulator such that this class $[P]$ is non-trivial.

Roe algebras

Consider a metric space X and let $\mathscr{H}$ be an ample X-module.
The *Roe algebra of X* is

$$C^*(X) = \overline{\left\{T \in \mathrm{B}(\mathscr{H}) \mid T \text{ locally compact, of finite propagation}\right\}}$$

For $Y \subseteq X$, the *Roe algebra localized at Y* is

$$C^*(Y \subseteq X) = \overline{\left\{T \in C^*(X) \mid T \text{ supported near } Y\right\}}.$$

Roe algebras are coarsely invariant!

What about boundary behavior?

Bulk-boundary correspondence

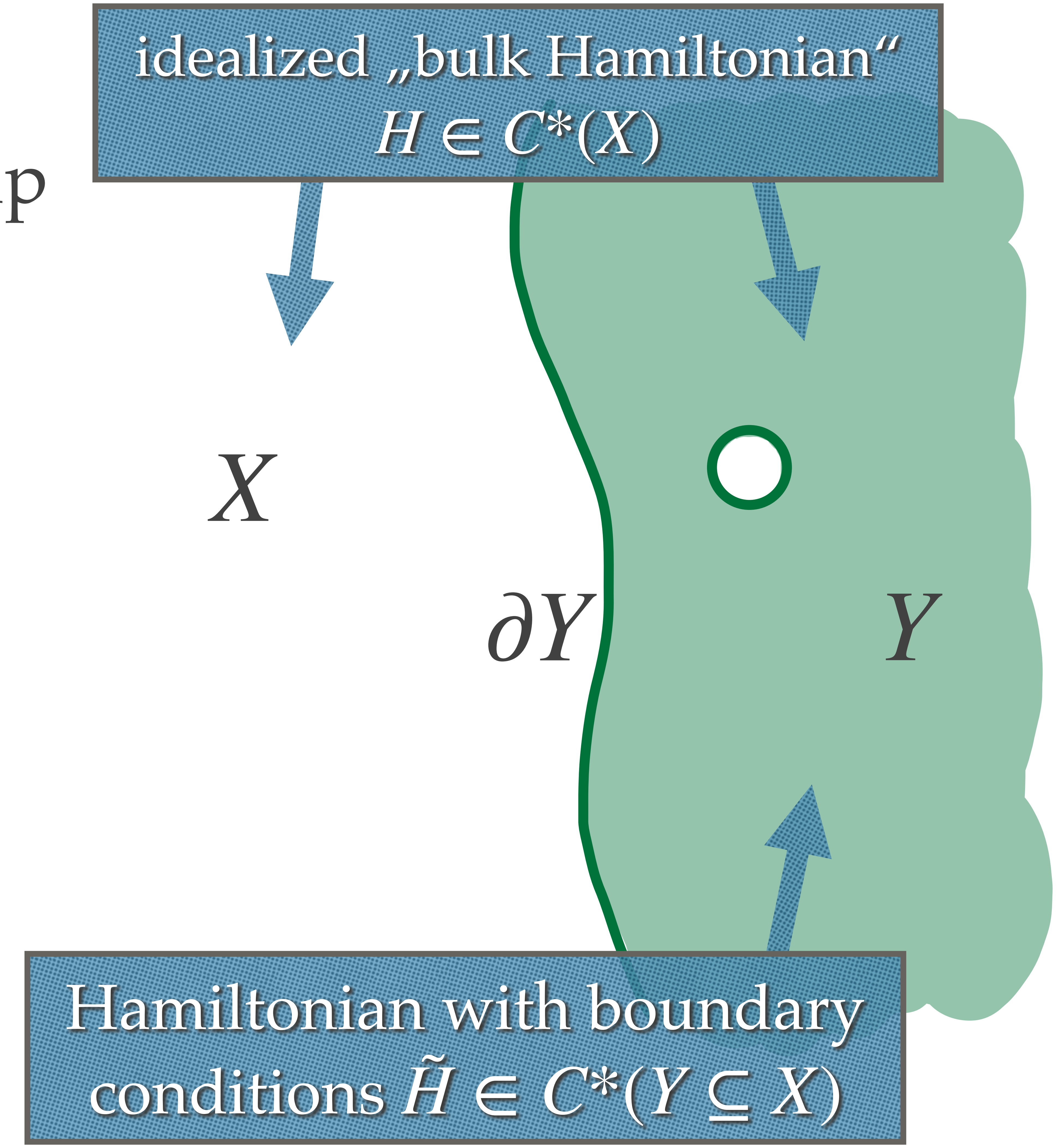
For suitable subsets $Y \subseteq X$, we have a *coarse Mayer-Vietoris sequence* and a corresponding boundary map

$$\delta : K_0(C^*(X)) \longrightarrow K_1(C^*(\partial Y \subseteq X))$$

Suppose that H is insulating at E and let P be the corresponding Fermi projection.

Theorem. If $\delta([P]) \neq 0$, then the Hamilton with boundary conditions $\tilde{H}$ has no spectral gap at E.

$\Longrightarrow$ conducting edge modes!



Proof.

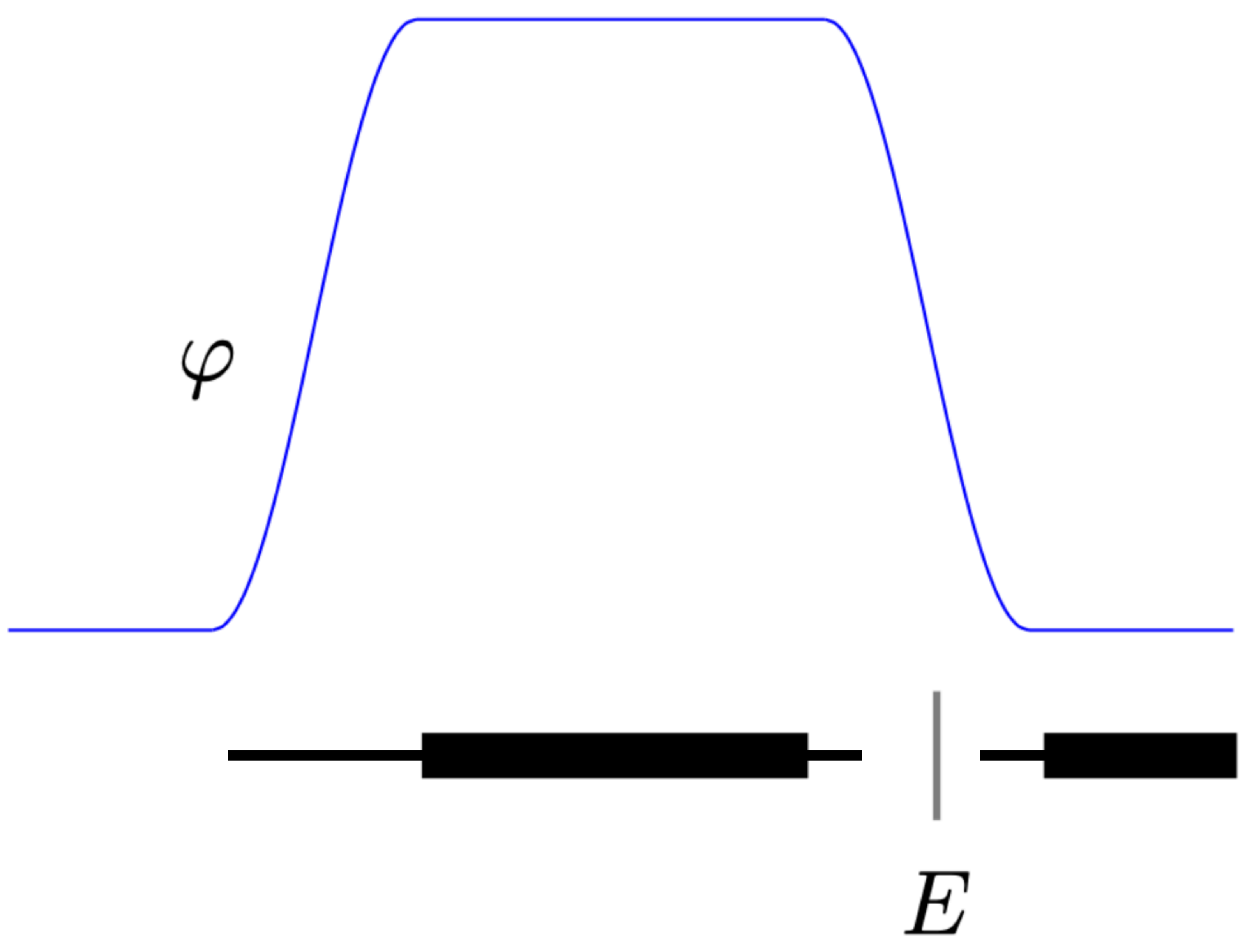
$$[P] \in K_0(C^*(X)) \overset{\delta}{\dashrightarrow} K_1(C^*(\partial Y))$$

$$\downarrow \pi_*$$

$$\underset{K_0(C^*(Y))}{\overset{[\tilde{P}]}{\Cap}} \longrightarrow K_0(C^*(Y)/C^*(\partial Y)) \xrightarrow[\text{Exp}]{} K_1(C^*(\partial Y))$$

Theorem. If $\delta([P]) \neq 0$, then the Hamilton with boundary conditions $\tilde{H}$ has no spectral gap at E.

Measures failure of $P \in C^*(Y)/C^*(\partial Y)$ to lift to a projection in $C^*(Y)$



If spectral gap was not filled

$\Longrightarrow$ can lift $P = \varphi(H)$ to $\tilde{P} = \varphi(\tilde{H})$ in $C^*(Y)$

Proof.

$$[P] \in K_0(C^*(X)) \xrightarrow{\;\delta\;} K_1(C^*(\partial Y))$$

$$\downarrow \pi_*$$

$$\underset{K_0(C^*(Y))}{\overset{[\tilde{P}]}{\Cap}} \longrightarrow K_0(C^*(Y)/C^*(\partial Y)) \xrightarrow[\text{Exp}]{} K_1(C^*(\partial Y))$$

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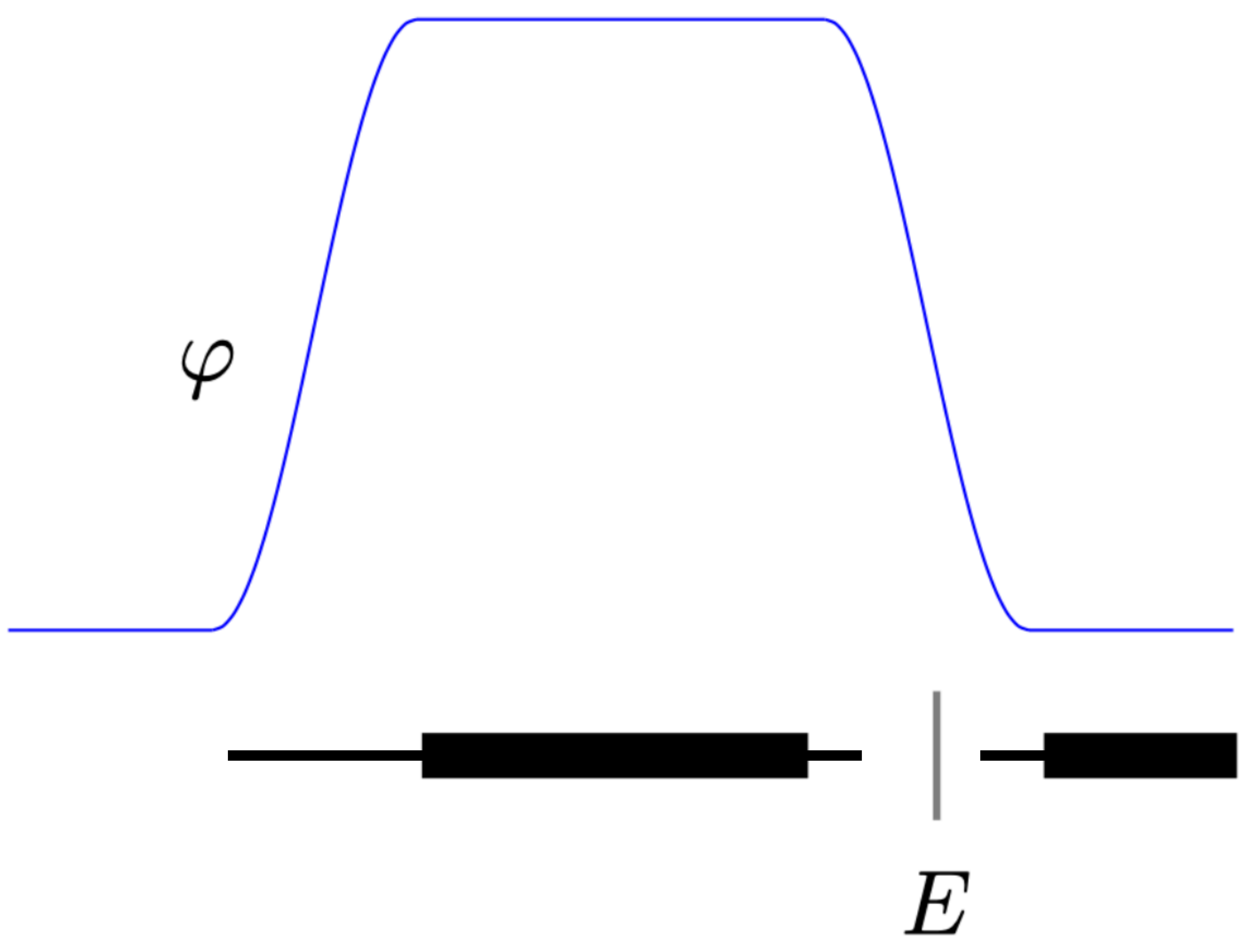


If spectral gap was not filled

$\implies$ can lift $P = \varphi(H)$ to $\tilde{P} = \varphi(\tilde{H})$ in $C^*(Y)$

$\implies \text{Exp}(\pi_*[P]) = 0 \implies \delta([P]) = 0$ $\square$

Example: Suppose that Y is a „coarse half space“ in $X = \mathbb{R}^2$.

- $K_0(C^*(X)) \cong \mathbb{Z}$
- Y and Y^c are flasque, so $K_\bullet(C^*(Y)) = K_\bullet(C^*(Y^c)) = 0$

$$\begin{array}{ccccc}
K_0(C^*(\partial Y)) & \longrightarrow & K_0(C^*(Y)) \oplus K_0(C^*(Y^c)) & \longrightarrow & K_0(C^*(X)) \\
\delta \uparrow & & & & \downarrow \delta \\
K_1(C^*(X)) & \longleftarrow & K_1(C^*(Y)) \oplus K_1(C^*(Y^c)) & \longleftarrow & K_1(C^*(\partial Y))
\end{array}$$

$\Longrightarrow$ boundary maps are isomorphisms.

$\Longrightarrow$ Landau Hamiltonian on Y has all gaps filled!

Equivariant version

If a group Γ acts on X, then we may get an equivariant version:

$$\begin{array}{ccccc}
C^*(\partial Y \subseteq Y) & \longrightarrow & QC^*(Y)^\Gamma & \longrightarrow & C^*(X)^\Gamma \\
\| & & \downarrow & & \downarrow \\
 & & & & C^*(X) \\
 & & & & \downarrow \\
C^*(\partial Y \subseteq Y) & \longrightarrow & C^*(Y) & \longrightarrow & C^*(Y)/C^*(\partial Y \subseteq Y)
\end{array}$$

quasi-invariant Roe algebra

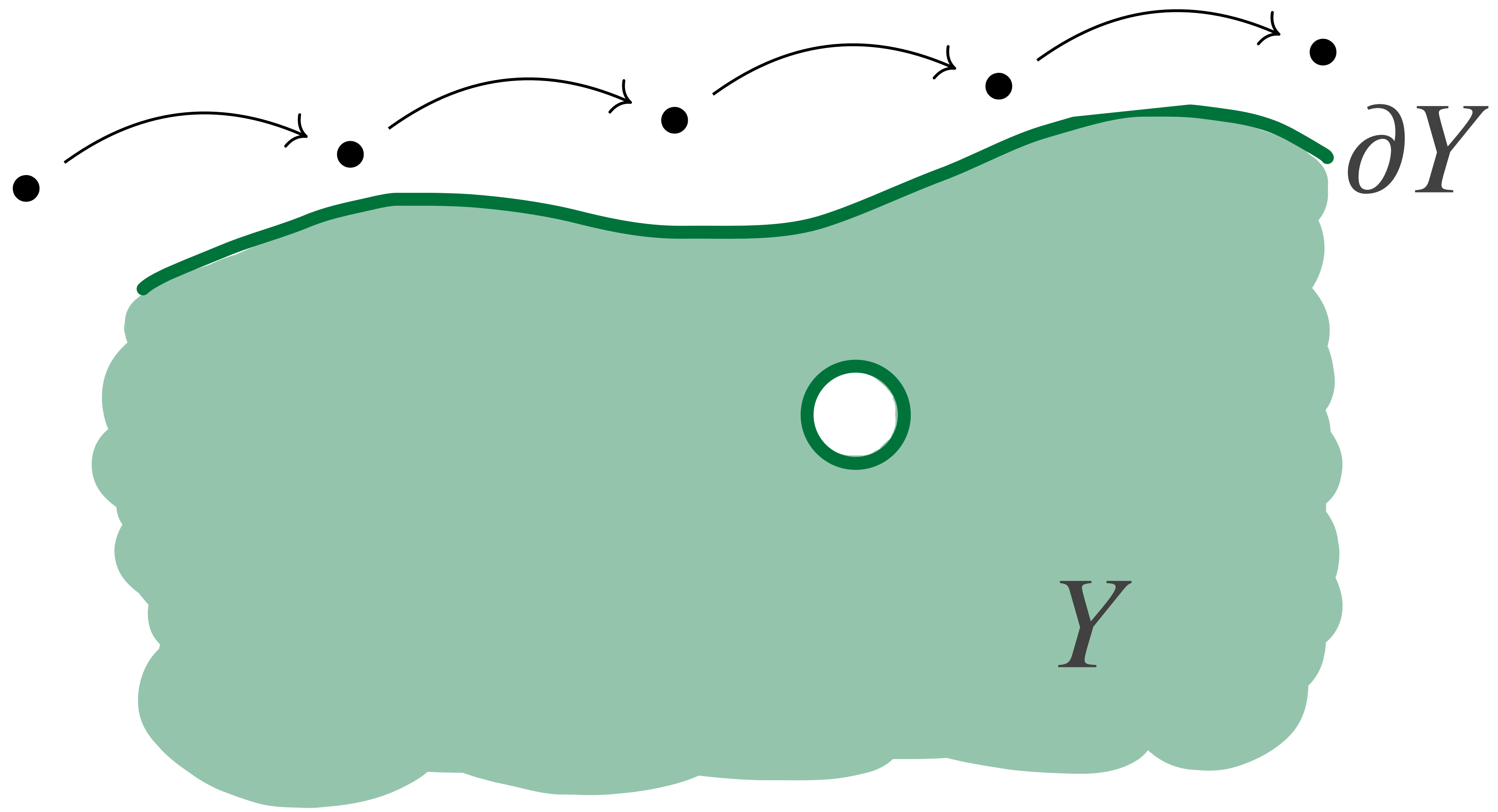
Cobordism invariance of topological edge-following states (2023), with G. C. Thiang
Breaking symmetries for equivariant coarse homology theories (2024), with Uli

Why do the edge states travel?

$$C^*_{\text{group}}(\mathbb{Z}) \otimes \mathbb{K} \cong C^*(\mathbb{Z})^{\mathbb{Z}} \subseteq C^*(\mathbb{Z})$$

Group C^*-algebra

inclusion induces isomorphism in K-theory



The group $K_1(C^*_{\text{group}}(\mathbb{Z})) \cong \mathbb{Z}$ is generated by the shift operator.

How to extract numbers from K-theory classes?

Kitaev's formula

$$\nu(P) = 12\pi i \sum_{x\in A} \sum_{y\in B} \sum_{z\in C} \left(P_{xy}P_{yz}P_{zx} - P_{xz}P_{zy}P_{yx}\right)$$

Idea: This defines a coarse cohomology class.

Quantization of conductance and the coarse cohomology of partitions (2024), with G. C. Thiang

Coarse cohomology

A *coarse cochain* is a locally bounded Borel function $\varphi : X^{n+1} \to \mathbb{R}$ such that $\operatorname{supp}(\varphi) \cap B_r(\Delta)$ is bounded for each $r > 0$.

The differential is the Alexander-Spanier differential

$$\delta\varphi(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

$\Longrightarrow$ get a cohomology groups $H\mathscr{X}^n(X)$.

Pairing with K-theory

algebra of finite propagation, locally trace-class operators

There is a map (the *Connes character map*)

$$\chi : H\mathscr{X}^n(X) \longrightarrow HC^n(\mathscr{B}(X)),$$

such that $\chi(f_0 \otimes \cdots \otimes f_n)(A_0, \ldots, A_n) = \mathrm{Tr}\big(f_0 A_0 f_1 A_1 \cdots f_n A_n\big).$

Composing with the cyclic homology Chern character gives pairings

$$K_0(\mathscr{B}(X)) \times H\mathscr{X}^{2n}(X) \longrightarrow \mathbb{R}$$

given by

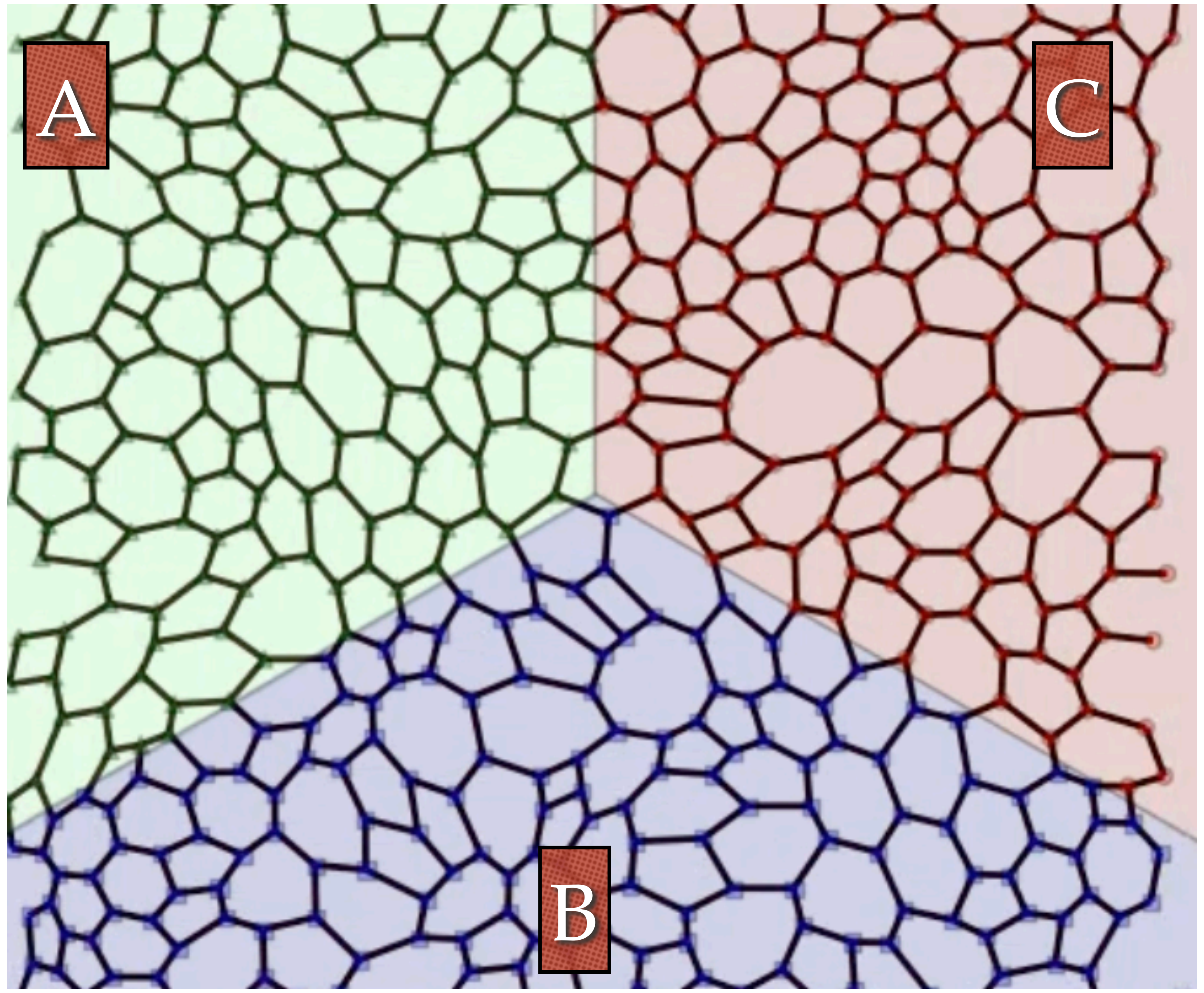
$$\langle [P], f_0 \otimes \cdots \otimes f_{2n} \rangle = \mathrm{Tr}\big(f_0 P f_1 P \cdots f_{2n} P\big).$$

Partition classes

Given a partition such that for all $r > 0$, $B_r(A) \cap B_r(B) \cap B_r(C)$ is bounded, we get a closed coarse 2-cochain by

$$\varphi_{A,B,C} = A \otimes B \otimes C + B \otimes C \otimes A + C \otimes A \otimes B$$
$$- A \otimes C \otimes B - C \otimes B \otimes A - B \otimes A \otimes C$$

Indicator functions



Partition classes

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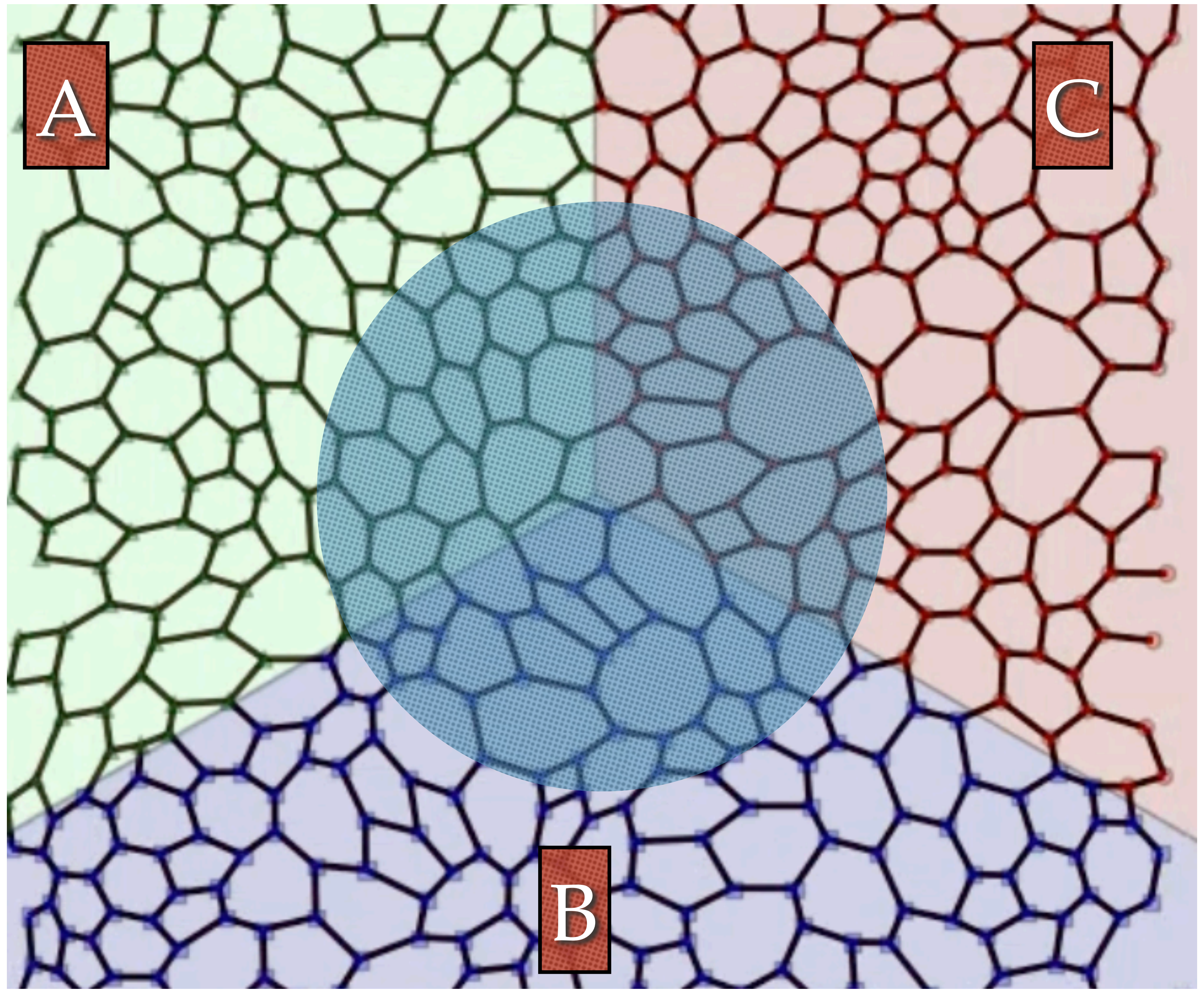


$$\varphi_{A,B,C} = A \otimes B \otimes C + B \otimes C \otimes A + C \otimes A \otimes B$$
$$- A \otimes C \otimes B - C \otimes B \otimes A - B \otimes A \otimes C$$

Indicator functions

Observe: For any $r > 0$, if $x = (x_1, x_2, x_3) \in \mathrm{supp}(\varphi_{A,B,C})$ is such that $d(x_i, x_j) \leq r$, then $x_1, x_2, x_3 \in B_r(A) \cap B_r(B) \cap B_r(C)$.

Partition classes

For pairing with K-theory, we get

$$\begin{aligned}\langle \varphi_{A,B,C}, P\rangle &= 3\big(\mathrm{Tr}(APBPCP) - \mathrm{Tr}(APCPBP)\big)\\ &= 3\sum_{x\in A}\sum_{y\in B}\sum_{z\in C}\big(P_{xy}P_{yz}P_{zx} - P_{xz}P_{zy}P_{yx}\big).\end{aligned}$$

This is (up to a factor of $4\pi i$) precisely the formula of Kitayev.

Note: This is well-defined, as if P has propagation bound r, then $APBPCP$ is supported on the bounded set $B_{3r}(A) \cap B_{3r}(B) \cap B_{3r}(C)$, hence trace-class.

Locality

Calculate: $$\mathrm{Tr}(APBPCP) = \sum_{x \in A} \mathrm{Tr}(xPBPCP) = \sum_{x \in A} \mathrm{Tr}(PBPCPx)$$

If P has propagation bound r, then xP has support in $B_r(x)$.

$$\implies \text{If } d(x, B) > r, \text{ then } xPB = 0 \text{ and if } d(x, C) > r, \text{ then } CPx = 0.$$

The same argument for B, C instead of A shows that $\langle \varphi_{A,B,C}, P \rangle$ can be calculated „near the coarse intersection" of A, B, C:

$$\langle \varphi_{A,B,C}, P \rangle = 3 \sum_{(x,y,z) \in B_r(A) \cap B_r(B) \cap B_r(C)} \left(P_{xy} P_{yz} P_{zx} - P_{xz} P_{zy} P_{yx} \right)$$

Locality

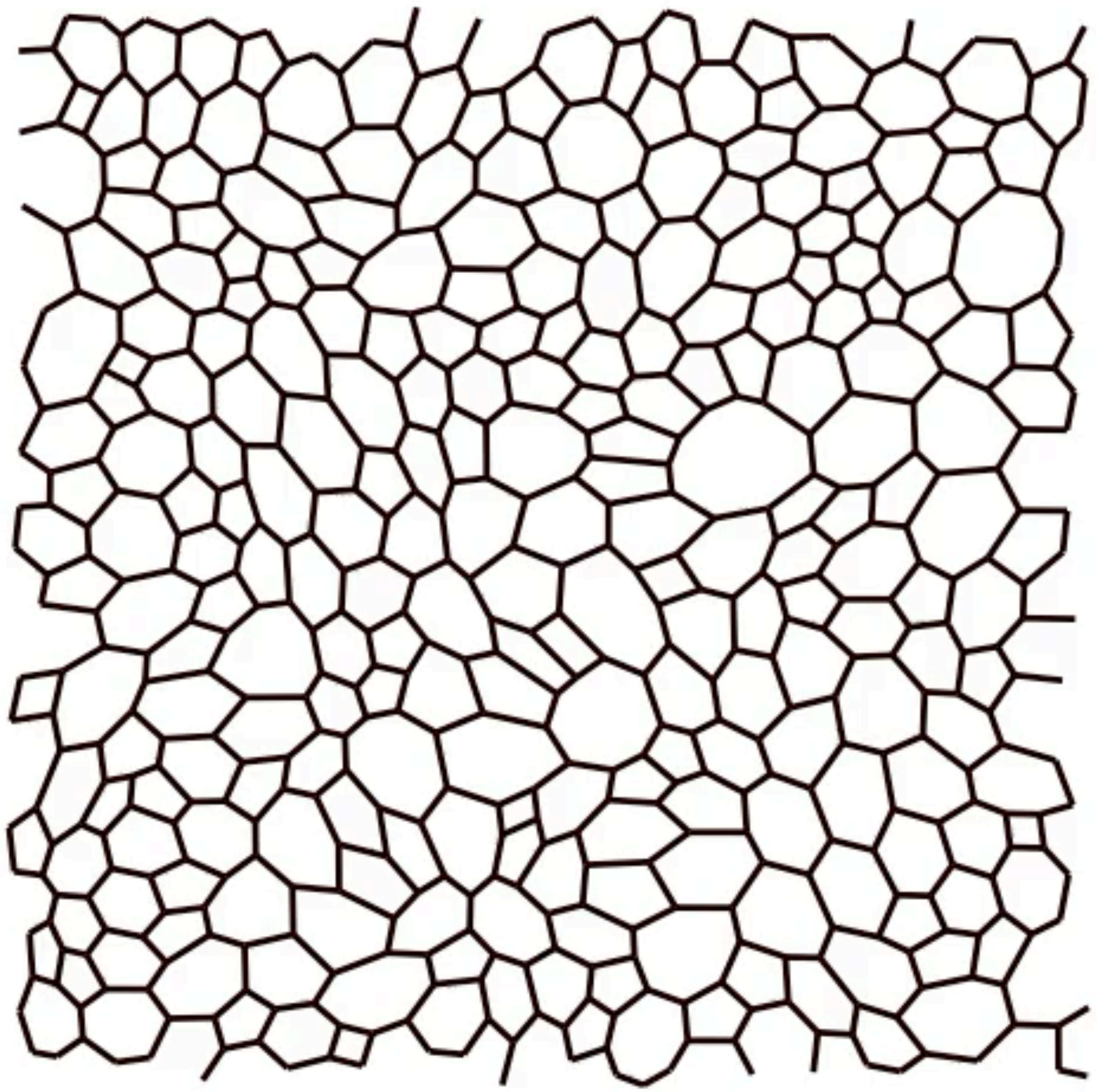
Strict locality fails if P does not have finite propagation, but remains *approximately* true if P has rapid decay.

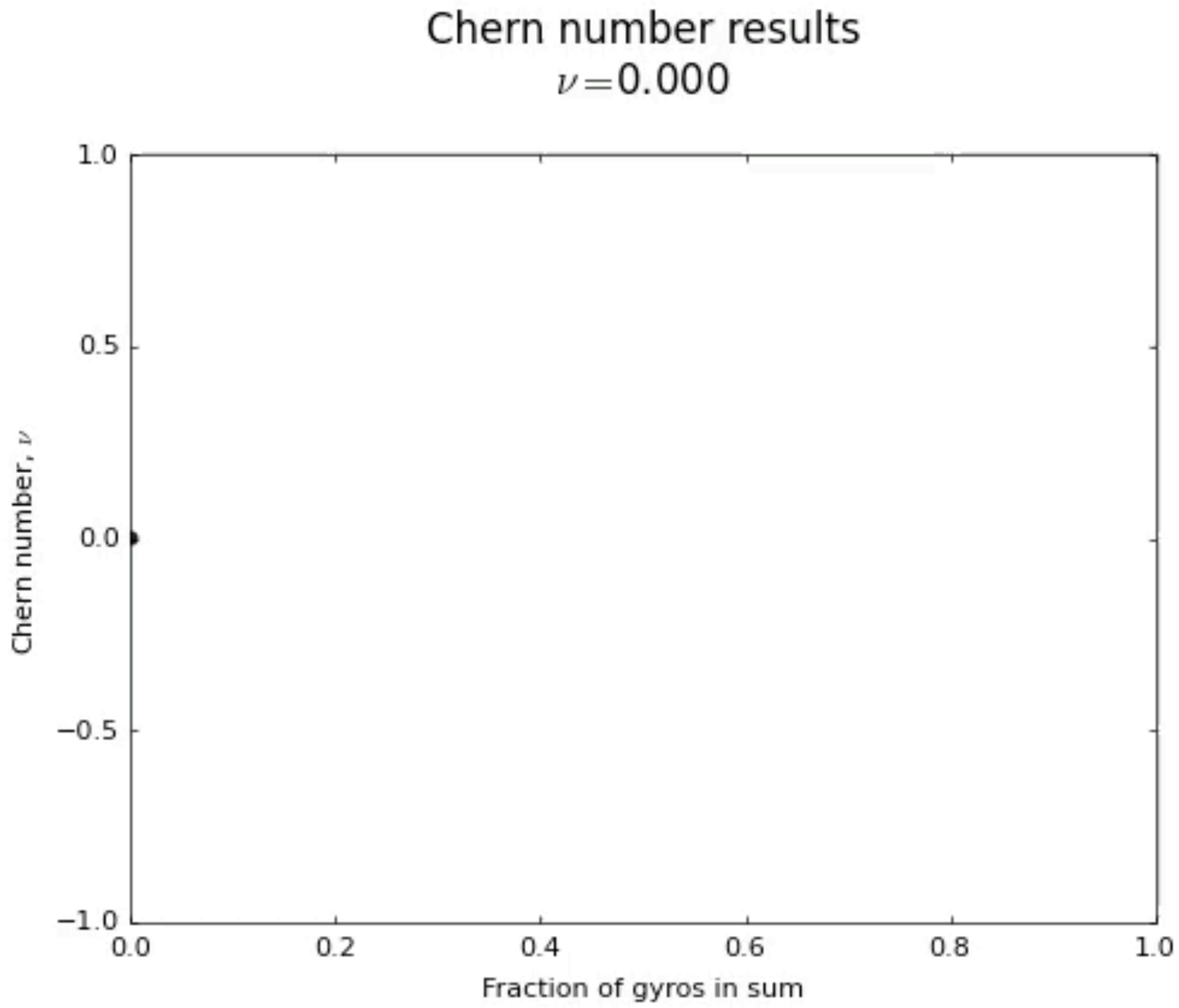
Example: If $Y \subseteq X$ and $\tilde{H}$ is the restriction of the Hamiltonian H to Y, equipped with boundary conditions, then by finite propagation of H, we have that

$$\tilde{P} - P = \varphi(\tilde{H}) - \varphi(H)$$

is *small* away from ∂Y, for any $\varphi \in C_0(\mathbb{R})$.

Real-space Chern number calculation for an amorphous network





Integrality?

Why is $4\pi i \cdot \langle \varphi_{A,B,C}, P \rangle \in \mathbb{Z}$?

Higson corona

For $r > 0$, the *r-variation* of $f : X \to \mathbb{C}$ is

$$\mathrm{Var}_r f(x) = \sup_{y \in B_r(x)} |f(x) - f(y)|.$$

Let

$$C_h(X) = \left\{ f \in C_b(X) \mid \forall r > 0 : \mathrm{Var}_r f \in C_0(X) \right\}$$

be the algebra of functions with bounded variation at infinity.

The *Higson corona* ∂X of X is the compact Hausdorff space such that

$$C(\partial X) = C_h(X)/C_0(X).$$

Higson corona

For $T \in C^*(X)$, we have that

- $[T, f]$ is compact if $f \in C_h(X)$;
- fT and Tf are compact if $f \in C_0(X)$.

Get a *-homomorphism

$$C(\partial X) \otimes C^*(X) \longrightarrow C^*(X)/\mathbb{K},$$

Which induces a map on K-theory,

$$K^{i+1}(\partial X) \times K_i(C^*(X)) \longrightarrow K_1(C^*(X)/\mathbb{K}) \xrightarrow{\partial} K_0(\mathbb{K}) \cong \mathbb{Z}$$

Have a commutative diagram

$$
\begin{array}{ccc}
K^1(\partial X) & \longrightarrow & \mathrm{Hom}\big(K_0(C^*(X)), \mathbb{Z}\big) \\
\Big\downarrow \mathrm{ch} & & \Big\downarrow \\
H^{\mathrm{odd}}(\partial X) & & \\
\Big\downarrow \tau & & \\
H\mathcal{X}^{\mathrm{ev}}(X) & \longrightarrow & \mathrm{Hom}\big(K_0(C^*(X)), \mathbb{R}\big)
\end{array}
$$

One may show that $4\pi i \cdot \varphi_{A,B,C}$ is the image of a generator of $K^1(\partial X)$ under the left vertical map. Hence $4\pi i \cdot \varphi_{A,B,C}$ pairs integrally with K_0.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