Primary and Secondary invariants of Dirac operators on G-proper manifolds

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Outline

Introduction

Higher indices on G-proper manifolds (primary invariants)

Rho invariants (secondary invariants)

Results

Equivariant index theory: a bit of history

- Atiyah, Bott, Segal and Singer extended the Atiyah-Singer index theorem to the case in which a compact Lie group G acts on the given compact manifold X and the relevant operator D is G-equivariant (commutes with the action)
- many subsequent contributions
- Atiyah was the first to pass to a non-compact situation by considering the universal cover $\widetilde{X}^{2\ell}$ of $X^{2\ell}$ compact
- $\Gamma := \pi_1(X)$ acts properly, freely and cocompactly on $\widetilde{X}$ and $\widetilde{X}/\Gamma = X$
- Atiyah defined a von Neumann index for a Γ-equivariant Dirac operator $\widetilde{D}$, $\mathrm{ind}_{\mathrm{vN}}(\widetilde{D}^+)$, and proved that it was equal to the Fredholm index of the operator D induced on X
- Connes and Moscovici (Topology, 1990), using K-theory and cyclic cohomology, extended Atiyah's theorem, defining higher indices $\mathrm{Ind}_\alpha(\widetilde{D}^+)$, $\alpha \in H^*(\Gamma)$ and proving explicit formulas

- there are (non-trivial) extensions of these results to manifolds with boundary (Donnelly, Lott, Leichtnam-P, Wahl, P-Schick-Zenobi, Chen-Wang-Xie-Yu); in this case the higher indices are parametrized by $\alpha \in HC^*(\mathbb{C}\Gamma, \langle g \rangle)$ with $\langle g \rangle$ an arbitrary conjugacy class. Need assumptions on Γ.
- Atiyah-Schmidt and Connes-Moscovici (Annals of Mathematics, 1982) extended Atiyah to non-compact Lie groups, considering the von Neumann index on G/K, with G a unimodular non-compact Lie group and K a maximal compact subgroup
- Hang Wang extended the result of Connes-Moscovici to an arbitrary G-proper manifold (will give definitions)
- we are interested in generalizing the results for coverings to results in G-equivariant index theory with G a non compact Lie group, both on manifolds without boundary and manifolds with boundary
- thus we consider equivariant index theory on G-proper manifolds, with G a non-compact Lie group.

G-Proper manifolds: gometric set up

We consider:

- G a connected linear real reductive Lie group
- $K < G$ maximal compact subgroup
- (X, h), a cocompact G-proper manifold, $\dim X$ even, $\partial X = \emptyset$, with a G-invariant riemannian metric h
- proper: the map $G \times X \to X \times X$, $(g, x) \to (x, gx)$ is proper
- Slice theorem: there exists a K-invariant compact submanifold $S \subset X$ s.t. the action map $[g, s] \to gs$, $g \in G$, $s \in S$, defines a G-equivariant diffeomorphism $G \times_K S \xrightarrow{\alpha} X$
- D, a $\mathbb{Z}_2$-graded odd G-equivariant Dirac operator on X acting on the sections of a G-equivariant vector bundle $E = E^+ \oplus E^-$

A warning

- let us consider $X := G \times_K S$
- on the left there is a "product" Dirac operator D_{split}
- it is stated in some published papers (not by me !) that $D_{\mathrm{split}} = D$
- this is wrong !
- several proofs in recent literature are affected by this mistake....
- in particular large and short time behaviour of $\exp(-tD^2)$.....

K_0-group

- Let A be a unital algebra, for us a C^*-algebra
- $K_0(A) = \{$formal differences of iso. classes of finitely generated projective A-modules$\}$ (Grothendieck group associated to the semigroup $\mathrm{Proj}(A)$)
- a fin. gen. proj. A-module is by definition a direct summand of a free A-module of finite rank: $A \oplus \cdots \oplus A$ (n times, for some n)
- thus a finitely generated projective A-module is the image of a projector in $M_{k\times k}(A)$ for some k
- so, an element in $K_0(A)$ is a formal difference of projectors in $M_{k\times k}(A)$, $k \in \mathbb{N}$
- if A is not unital we work with its unitalization
- if $\mathcal{A} \subset A$ is a subalgebra which is dense and holomorphically closed then $K_0(\mathcal{A}) = K_0(A)$

G-Proper manifolds: compactly supported index classes

- One has a G-equivariant pseudodifferential calculus with G-compact support $\Psi^*_{G,c}(X,E)$
- Can use this calculus to show that D^+ has a parametrix $Q \in \Psi^{-1}_{G,c}(X,E^-,E^+)$ with remainders $S_\pm \in \Psi^{-\infty}_{G,c}$
- $\Psi^{-\infty}_{G,c}(X,E)$ is the algebra of G-equivariant smoothing kernels of G-compact support.
- We set $\mathcal{A}^c_G(X,E) := \Psi^{-\infty}_{G,c}(X,E)$
- We define the projector

$$P := \begin{pmatrix} S_+^2 & S_+(I+S_+)Q \\ S_-D^+ & I-S_-^2 \end{pmatrix}.$$

- The compactly supported index class is by definition $\mathrm{Ind_c}(D) := [P]-[e_1] \in K_0(\mathcal{A}^c_G(X,E))$ with $e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

We shall often forget about vector bundles in the notation...

G-Proper manifolds: C^*-index classes

- we now consider $C^*(X,E)^G :=$ closure of $\Psi^{-\infty}_{G,c}(M,E)$ in $\mathcal{B}(L^2(X,E))$
- this is called the Roe algebra;
- $\mathrm{Ind}(D) =: [P] - [e_1] \in K_0(C^*(X,E)^G)$ is the C^*-index class
- Important: $\mathrm{Ind}(D) \in K_0(C^*(X,E)^G)$ is the class we are interested in !
- Indeed: (i) C^*-index class of the spin-Dirac operator of a G-invariant PSC metric vanishes
- (ii) C^*-index class of the signature operator is a G-homotopy invariant
- Remark: we can give a different representative of this index class, the Connes-Moscovici projector:

$$V_{\mathrm{CM}}(D) = \begin{pmatrix} e^{-D^-D^+} & e^{-\frac{1}{2}D^-D^+}\left(\frac{I-e^{-D^-D^+}}{D^-D^+}\right)D^- \\ e^{-\frac{1}{2}D^+D^-}D^+ & I - e^{-D^+D^-} \end{pmatrix}$$

Cyclic cohomology

- $\mathcal{A}$ = Fréchet algebra over $\mathbb{C}$
- Hochschild cochains of degree k: $C^k(\mathcal{A})$
- $C^k(\mathcal{A})$: all continuous $k+1$-linear functionals Φ on $\mathcal{A}$
- Hochschild codifferential $b\colon C^k(\mathcal{A}) \to C^{k+1}(\mathcal{A})$
  $$b\Phi(a_0 \otimes \cdots \otimes a_{k+1}) = \sum_{i=0}^{k}(-1)^i\Phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) + (-1)^{k+1}\Phi(a_{k+1}a_0 \otimes a_1 \otimes \cdots \otimes a_k).$$
- Hochschild cohomology of $\mathcal{A}$ is cohomology of $(C^*(\mathcal{A}), b)$
- a Hochschild k-cochain $\Phi \in C^k(\mathcal{A})$ is called *cyclic* if $\Phi(a_k, a_0, \ldots, a_{k-1}) = (-1)^k\Phi(a_0, a_1, \ldots, a_k)$
- $C^k_\lambda(\mathcal{A})$= {cyclic cochains} ; it is closed under b.
- cyclic cohomology $HC^*(A)$ = cohomology of $(C^k_\lambda(\mathcal{A}), b)$.

0-cyclic cocycles (traces)

- $HC^0(\mathcal{A}) = \{\Phi : \mathcal{A} \to \mathbb{C} \text{ continuous } \mid \Phi(a_0 a_1) = \Phi(a_1 a_0)\}$
- $\Rightarrow HC^0(\mathcal{A}) =$ continuous traces on $\mathcal{A}$
- we remark that there exists a natural pairing $\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \otimes HC^0(\mathcal{A}) \longrightarrow \mathbb{C}$:

$$\langle (p_{ij}), \Phi \rangle := \sum \Phi(p_{jj})$$

- in general there is a paring $\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \otimes HC^{2\mathrm{k}}(\mathcal{A}) \longrightarrow \mathbb{C}$

$$\langle [P], \Phi \rangle = \frac{1}{k!} \sum_{i_0, i_1, \ldots i_{2k}} \Phi(p_{i_0 i_1}, \ldots, p_{i_{2k} i_0})$$

Higher numeric indices

- we want to extract numbers out of the C^*-index class $\mathrm{Ind}(D) \in K_0(C^*(X)^G)$
- first we extract numbers out of the compactly supported index class $\mathrm{Ind}_c(D) \in K_0(\mathcal{A}_G^c(X))$
- we use the cyclic cohomology groups $HC^*(\mathcal{A}_G^c(X))$
- and the pairing $\langle \cdot, \cdot \rangle \; : K_0(\mathcal{A}_G^c(X)) \otimes HC^{\mathrm{even}}(\mathcal{A}_G^c(X)) \longrightarrow \mathbb{C}$
- $\Rightarrow$ want cyclic cocycles for the algebra $\mathcal{A}_G^c(X)$
- next we want to
  (1) find a dense and holomorphically closed subalgebra $\mathcal{A}_G^\infty(X)$ of $C^*(X)^G$ such that

$$\mathcal{A}_G^c(X) \subset \mathcal{A}_G^\infty(X) \subset C^*(X)^G$$

  (this will ensure that $K_0(\mathcal{A}_G^\infty(X)) = K_0(C^*(X)^G)$) and
  (2) EXTEND the cyclic cocycles from $\mathcal{A}_G^c(X)$ to $\mathcal{A}_G^\infty(X)$
- the latter step, which is analytical, is often quite technical

Cyclic cocycles from group cocycles

- we start with cyclic cocycles for $C_c^\infty(G)$
- we first define $H^*_{\mathrm{diff}}(G)$ and prove that there is a morphism $H^*_{\mathrm{diff}}(G) \to HC^*(C_c^\infty(G))$
- let's define $H^*_{\mathrm{diff}}(G)$: we consider $C^\infty(G^{\times k})$; $\delta : C^\infty(G^{\times k}) \to C^\infty(G^{\times k})$ defined by $\delta(\varphi)(g_1, \cdots, g_{k+1}) = \varphi(g_2, \cdots, g_{k+1}) - \varphi(g_1 g_2, \cdots, g_{k+1}) + \cdots + (-1)^k \varphi(g_1, \cdots, g_k g_{k+1}) + (-1)^{k+1} \varphi(g_1, \cdots, g_k)$.
- differentiable group cohomology $H^*_{\mathrm{diff}}(G)$ is the cohomology of $(C^\infty(G^{\times *}), \delta)$
- there is a Van Est isomorphism $H^*_{\mathrm{diff}}(G) = H^*_{\mathrm{inv}}(G/K)$

Cyclic cocycles from group cocycles (cont)

- let $\varphi \in H^k_{\text{diff}}(G)$
- let $a_0, \ldots, a_k \in C_c^\infty(G)$

Define $\tau_\varphi(a_0, \ldots, a_k) :=$
$\int_{G^{\times k}} \varphi(e, g_1, \ldots, g_1 \cdots g_k) a_0((g_1 \cdots g_k)^{-1}) a_1(g_1) \cdots a_k(g_k) dg_1 \cdots dg_k$

Proposition: $\tau_\varphi \in HC^*(C_c^\infty(G))$

Conclusion: we have defined a map
$H^*_{\text{diff}}(G) \ni \varphi \longrightarrow \tau_\varphi \in HC^*(C_c^\infty(G))$
which is a morphism

Example

- we take $G = SL(2, \mathbb{R})$
- observe that $K = SO(2)$ is a max compact subgroup and that $SL(2, \mathbb{R})/SO(2) = \mathbb{H}^2$, the hyperbolic plane;
- $gSO(2) := \overline{g}$
- define an element $\omega \in H^2_{\mathrm{diff}}(SL(2, \mathbb{R}))$ by $\omega(g_0, g_1, g_2) = \mathrm{Area}_{\mathbb{H}^2} \Delta(\overline{g}_0, \overline{g}_1, \overline{g}_2)$ (geodesic triangle)
- τ_ω is the following cyclic 2-cocycle on $C_c^\infty(SL(2, \mathbb{R}))$:

$$\tau_\omega(f_0, f_1, f_2) :=$$

$$\int_{SL(2,\mathbb{R})^{\times 2}} f_0((g_1 g_2)^{-1}) f_1(g_1) f_2(g_2) \mathrm{Area}_{\mathbb{H}^2}(\Delta(\overline{e}, \overline{g}_1, \overline{g}_2)) dg_1 dg_2$$

Cyclic cocycles from orbital integrals

- Given $g \in G$ semisimple we consider the so called orbital integral tr_g: if $Z := Z_G(g)$ and $f \in C_c^\infty(G)$ then

$$\mathrm{tr}_g(f) := \int_{G/Z} f(xgx^{-1}) d(xZ) \,.$$

- Proposition: tr_g is a trace and hence defines a class $\mathrm{tr}_g \in HC^0(C_c^\infty(G))$
- tr_g does not come from the differentiable cohomology of G; it is a delocalized cocycle

Cyclic cocycles from higher orbital integrals

- Song and Tang defined for each $P < G$ cuspidal parabolic subgroup with Langlands decomposition $P = MAN$, $m := \dim A$ and $g \in M$ semisimple an element

$$\Phi_g^P \in HC^m(C_c^\infty(G))$$

- these are higher delocalized cocycles
- we skip the definition which is somewhat involved

Cyclic cocycles for $\mathcal{A}^c_G(X,E)$

- given $\varphi \in H^n_{\mathrm{diff}}(G)$ we have defined $\tau_\varphi \in HC^n(C^\infty_c(G))$
- can similarly define $\tau^X_\varphi \in HC^n(\mathcal{A}^c_G(X))$.
- $\tau^X_\varphi(k_0,\dots,k_n)$ is

$$\int_{G^k}\int_{X^{(k+1)}} c(x_0)\cdots c(x_n)k_0(x_0,g_1x_1)\cdots k_n(x_n,(g_1\cdots \quad g_n)^{-1}x_0)$$
$$\varphi(e,g_1,g_1g_2,\dots,g_1\cdots g_n)dx_0\cdots dx_n dg_1\cdots dg_n.$$

c is a cut-off funct. on X: $\int_G c(g^{-1}x)dg = 1\ \forall x\in X$

- similarly, from tr_g we define $\mathrm{tr}^X_g \in HC^0(\mathcal{A}^c_G(X))$:

$$\mathrm{tr}^X_g(k) := \int_{G/Z}\int_X c(hgh^{-1}x)\mathrm{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}x,x))dx\, d(hZ)$$

- and similarly, from $\Phi^P_g \in HC^m(C^\infty_c(G))$ we can define $\Phi^P_{X,g} \in HC^m(\mathcal{A}^c_G(X))$

Extension of cyclic cocycles on G

- we have defined cyclic cocycles on $C_c^\infty(G)$
- we must fix a dense holomorphically closed subalgebra of $C_r^*(G)$
- the Harish-Chandra algebra $\mathcal{C}(G)$ is such a subalgebra
- $\mathcal{C}(G)$ is made of functions of "rapid decay" on G
- $C_c^\infty(G) \subset \mathcal{C}(G) \subset C_r^*(G)$
- Theorem: the cyclic cocycles τ_φ extend continuously to $\mathcal{C}(G)$ (P-Posthuma 2020)
- Theorem: the cyclic cocycle tr_g extends continuously to $\mathcal{C}(G)$ (Harish-Chandra)
- Theorem: the cyclic cocycles Φ_g^P extend continuously to $\mathcal{C}(G)$ (Song-Tang 2018)

Harish-Chandra smoothing operators

- go back to $\mathcal{A}^c_G(X) := \Psi^{-\infty}_{G,c}(X)$, the smoothing G-equivariant operators on X of G-compact support
- recall slice theorem: $X = G \times_K S$
- as a consequence $\mathcal{A}^c_G(X) = (C^\infty_c(G) \widehat{\otimes} \Psi^{-\infty}(S))^{K \times K}$
- define $\mathcal{A}^\infty_G(X) := (\mathcal{C}(G) \widehat{\otimes} \Psi^{-\infty}(S))^{K \times K}$
  these are the Harish-Chandra smoothing operators
- $\mathcal{A}^\infty_G(X)$ dense and holomorphically closed in $C^*(X)^G$
- from extension properties of τ_φ, tr_g, Φ^P_g $\Rightarrow$

$$\tau^X_\varphi\,, \quad \mathrm{tr}^X_g\,, \quad \Phi^P_{X,g} \text{ extend continuously to } \mathcal{A}^\infty_G(X)$$

Higher indices

- Theorem (P-Postuma 2020) The Connes-Moscovici projector

$$V_{\mathrm{CM}}(D) = \begin{pmatrix} e^{-D^-D^+} & e^{-\frac{1}{2}D^-D^+}\left(\frac{I-e^{-D^-D^+}}{D^-D^+}\right)D^- \\ e^{-\frac{1}{2}D^+D^-}D^+ & I - e^{-D^+D^-} \end{pmatrix}$$

  has entries in $\mathcal{A}_G^\infty(X)$
- this defines a "smooth" index class: $\mathrm{Ind}_\infty(D) \in K_0(\mathcal{A}_G^\infty(X)) = K_0(C^*(X)^G)$
- we define higher indices by pairing:

$$\langle \mathrm{Ind}_\infty(D), \tau_\varphi^X \rangle , \quad \langle \mathrm{Ind}_\infty(D), \mathrm{tr}_g^X \rangle , \quad \langle \mathrm{Ind}_\infty(D), \Phi_{X,g}^P \rangle$$

- there are, correspondingly, 3 index theorems

Higher index formulae

- Pflaum-Posthuma-Tang give a formula $\langle \mathrm{Ind}_\infty(D), \tau_\varphi^X \rangle$: if $\varphi \in H^{2p}_{\mathrm{diff}}(G)$

$$\mathrm{Ind}_\varphi(D) := \langle \mathrm{Ind}_\infty(D), \tau_\varphi^X \rangle = C(p) \int_X c\mathrm{AS}(X) \wedge \omega_\varphi$$

  with c a cut-off function, ω_φ an explicit closed G-invariant form on X.
- Pflaum-Posthuma-Tang use algebraic index theorem of Nest-Tsygan;
  P-Posthuma gave a heat kernel proof using Getzler rescaling

Higher genera

Consider the signature operator D^{sign} and, if X admits a G-invariant spin structure, the spin-Dirac operator D^{spin}.
Then up to the factor $C(p)$,

- $\mathrm{Ind}_{\varphi}(D^{\mathrm{sign}})$ equals $\int_X c\, L(X) \wedge \omega_{\varphi}$ with c a a cut-off funct.
- $\mathrm{Ind}_{\varphi}(D^{\mathrm{spin}})$ equals $\int_X c\, \widehat{A}(X) \wedge \omega_{\varphi}$
- define the higher signature associated to $[\varphi]$ as $\sigma(X, [\varphi]) := \int_X c\, L(X) \wedge \omega_{\varphi}$
- define the higher $\widehat{A}$ genus as $\widehat{A}(X, [\varphi]) := \int_X c\, \widehat{A}(X) \wedge \omega_{\varphi}$
- index class of the signature operator is a G-homotopy invariant (Fukumoto)
- index class of spin-Dirac operator vanishes if $\exists$ a G-invariant metric of positive scalar curvature (Guo-Mathai-Wang)
- thus, under our assumptions, $\sigma(X, [\varphi])$ is a G-homotopy invariant and $\widehat{A}(X, [\varphi])$ vanishes in the presence of PSC.

Delocalized (higher) index theorems

- Peter Hochs and Hang Wang proved the following index formula
$$\langle \mathrm{Ind}_\infty(D), \tau_g^X \rangle = \int_{X^g} c^g \mathrm{AS}_g(X)$$
- $\mathrm{AS}_g(X)$ is the Atiyah-Segal form on the fixed point set X^g
- warning: they use the wrong operator....; corrected proof in P-Postuma-Song-Tang (August 2023).
- Hochs-Song-Tang give a formula for $\langle \mathrm{Ind}_\infty(D), \Phi^P_{X,g} \rangle$ by a clever reduction to a 0-degree index theorem (à la Hochs-Wang) on the M-manifold X/AN with $P = MAN$:
$$\langle \mathrm{Ind}_\infty(D_Y), \Phi^P_{Y,g} \rangle = \int_{(Y_0/AN)_g} c^g_{Y_0/AN} \mathrm{AS}(Y_0/AN)_g$$

Questions

- what about G-proper manifolds with boundary ?
- Can we prove a higher (delocalized) Atiyah-Patodi-Singer index theorem for cocompact G-proper manifolds *with* boundary ?
- Can we define secondary invariants (higher rho numbers) for an invertible operator on a cocompact G-proper manifold *without* boundary by looking at the boundary correction term in these APS theorems ?
- Are these higher rho numbers interesting invariants ?

Short answers

- for the cyclic cocycles τ_φ, $\varphi \in H^*_{\mathrm{Diff}}(G)$ (and the corresponding τ^X_φ) a higher APS index theorem is proved by P-Posthuma (2021)
- for the delocalized 0-cyclic cocycle tr_g defined by the orbital integral a APS index theorem has been proved by Peter Hochs-Hang Wang-Bai Ling Wang (2020) under the assumption that $G/Z_G(g)$ is compact
- P-Posthuma-Song-Tang : completely different proof and no assumption on $G/Z_G(g)$ (August 2023)
- P-Posthuma-Song-Tang : extension to perturbed operators $D + A$, with A a Lafforgue smoothing operators (quite a technical proof, August 2023)
- for the delocalized m-cocycles Φ^P_g defined by higher orbital integrals this is again a result by P-Posthuma-Song-Tang
- crucial technique: interplay between absolute and relative K-theory and absolute and relative cyclic cohomology

Precise statements: geometric data

- Y_0 is a cocompact G-proper manifold with boundary
- metrics, bundles, connections etc are all of product type near the boundary
- D is a G-equivariant Dirac operator; D_∂ boundary operator
- Y is the G-manifold with cylindrical end associated to Y_0
- if D_∂ is L^2-invertible than there exists a well defined $\mathrm{Ind}_{C^*}(D) \in K_*(C^*(Y_0 \subset Y)^G)$ (John Roe)
- we want to define higher C^*-indices and prove higher C^* Atiyah-Patodi-Singer index formulas

Statements: higher APS indices

Theorem

(1) There exists a dense holomorphically closed subalgebra $\mathcal{A}^\infty_G(Y)$ *of* $C^*(Y_0 \subset Y)^G$*(defined in terms of the residual b-operators)*
(2) There exists a smooth representative $\mathrm{Ind}_\infty(D)$ *of the index class in* $K_0(\mathcal{A}^\infty_G(Y)) = K_0(C^*(Y_0 \subset Y)^G)$.
(3) The cyclic cocycles

$$\tau^Y_\varphi \quad \mathrm{tr}^Y_g\,, \quad \Phi^P_{Y,g}$$

are well defined in $HC^*(\mathcal{A}^\infty_G(Y))$
(4) by pairing we obtain higher APS indices

$$\langle \mathrm{Ind}_\infty(D), \tau^Y_\varphi\ \rangle\,, \quad \langle \mathrm{Ind}_\infty(D), \mathrm{tr}^Y_g\ \rangle\,, \quad \langle \mathrm{Ind}_\infty(D), \Phi^P_{Y,g}\rangle\,.$$

Statements: index formula for τ_g

- Because of time we skip the APS index theorem for $\langle \mathrm{Ind}_\infty(D), \tau^Y_\varphi \rangle$ (P-Posthuma, Annals of K-theory, 2021)
- we concentrate on $\langle \mathrm{Ind}_\infty(D), \mathrm{tr}^Y_g \rangle$ and $\langle \mathrm{Ind}_\infty(D), \Phi^P_{Y,g} \rangle$; these are delocalized (higher) APS indices
- we begin with $\langle \mathrm{Ind}_\infty(D), \mathrm{tr}^Y_g \rangle$

Theorem

Assume D_∂ L^2-invertible. Then the delocalized eta invariant

$$\eta_g(D_\partial) := \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{tr}^{\partial Y}_g (D_\partial \exp(-tD^2_\partial)) \frac{dt}{\sqrt{t}}$$

is well defined and

$$\langle \mathrm{Ind}_\infty(D), \mathrm{tr}^Y_g \rangle = \int_{(Y_0)^g} c^g \mathrm{AS}_g(Y_0) - \frac{1}{2} \eta_g(D_\partial) \,,$$

With Posthuma, Song and Tang we have spent **a lot of** energy proving the following result (where we do not assume that the operator is a boundary operator):

Theorem

Let (X, g) be a cocompact G-proper manifold **without** *boundary and let D be an L^2-invertible G-equivariant Dirac-type operator. Let g be a semi-simple element. Then the integral*

$$\eta_g(D) := \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{tr}_g^X(D \exp(-tD^2)) \frac{dt}{\sqrt{t}} \tag{1}$$

converges.

Perturbed operators

- In a second paper we also deal with $D_X + A_X$, $A_X \in \mathcal{A}^\infty_G(X)$, X without boundary and prove convergence of $\eta_g(D + A)$ under the assumption that $D_X + A_X$ is L^2-invertible; large time behaviour rather delicate
- we then prove a higher APS index theorem for operators $D_Y + A_Y$ with Y manifold with cylindrical ends and A_Y a lift of $A_{\partial Y}$ such that $D_{\partial Y} + A_{\partial Y}$ is invertible
- the first result applies for example to the disjoint union of two G-proper manifold without boundary X_1, X_2 for which there is a G-homotopy equivalence f : $X_1 \to X_2$
- second result applies to two G-manifolds with boundary Y_1, Y_2 with a G-homotopy equivalence f : $\partial Y_1 \to \partial Y_2$
- on Galois coverings these results are crucial in connection with the surgery exact sequence in differential topology (Wahl, P-Schick)

Statements: index formula for Φ_g^P

- Next we tackle $\langle \mathrm{Ind}_\infty(D), \Phi^P_{Y,g} \rangle$.

Theorem
Assume $D_{\partial Y}$ L^2-invertible. Then

$$\langle \mathrm{Ind}_\infty(D_Y), \Phi^P_{Y,g} \rangle = \int_{(Y_0/AN)_g} c^g_{Y_0/AN} \mathrm{AS}(Y_0/AN)_g - \frac{1}{2}\eta_g(D_{\partial(Y_0/AN)})$$

This is proved in P-Postuma-Song-Tang by jazzing-up to manifolds with boundary the reduction procedure of Hochs-Song-Tang and then applying the previous theorem.

rho numbers

- Let X a G-proper manifold without boundary. Assume we have a G-equivariant spin structure.
- if h is a G-invariant metric of positive scalar curvature then we can define
$$\rho_g(\mathsf{h}) = \eta_g(D_{\mathsf{h}})$$
- we can also define $\rho_g^P(\mathsf{h})$
- if g does not have fixed points then these are invariants for equivariant concordance and equivariant psc-bordism
- we could similarly define a rho number associated to a G-equivariant homotopy equivalence $\mathsf{f} : X \to X'$ by using the Fukumoto-Hilsum-Skandalis perturbation $A(\mathsf{f})$ and considering
$$\rho_g(\mathsf{f}) := \eta_g(D^{\mathrm{sign}} + A(\mathsf{F}))$$
- connection with G-h-cobordism through our APS theorem

Papers

- P.P. and Hessel B. Posthuma, *Higher genera for proper actions of Lie groups*, Ann. K-Theory 4 (2019)
- P.P. and Hessel B. Posthuma, *Higher genera for proper actions of Lie groups, II: the case of manifolds with boundary*, Ann. K-Theory 6 (2021)
- P.P., Hessel B. Posthuma, Yanli Song, and Xiang Tang, *Higher Orbital Integrals, Rho Numbers and Index Theory*, arXiv August 2023
- P.P., Hessel B. Posthuma, Yanli Song, and Xiang Tang, *Heat kernels of perturbed operators and index theory on G-proper manifolds*, arXiv August 2023
- P.P. and Xiang Tang, *Primary and secondary invariants of Dirac operators on G-proper manifolds* Proceedings of Symposia in Pure Mathematics, Volume 105, 311-351, 2023. Proceedings of the conference, *Cyclic cohomology at 40. Achievements and future prospects*.
  This is a survey !! Last version on arxiv is the best...

Thank You

Happy Birthday to Ulrich !!