Local index theory for the Rarita-Schwinger operator

Alberto Richtsfeld

University of Potsdam

From Analysis to Homotopy Theory,
Greifswald
13-17 May 2024

References

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Geometric structures

Let $\mathbf{Man}_n$ be the category of compact, connected, smooth n-manifolds with local diffeomorphisms as morphisms.

- Let $G_n \in \{\mathrm{O}(n), \mathrm{SO}(n), \mathrm{Pin}(n), \mathrm{Spin}(n)\}$.
- There is a subcategory $G_n - \mathbf{Man}_n$ of $\mathbf{Man}_n$, whose objects are the manifolds admitting G_n-structures.
- There is a functor

$$G_n - \mathrm{Str} : G_n - \mathbf{Man}_n^{op} \to \mathbf{Set},$$

  sending a manifold M to the set of different G_n-structures on M.

Geometric structures

The set of orientations $\mathcal{O}(M)$ on M is given by $\pi_0(\tilde{M})$, where $\tilde{M}$ is the orientation double cover of M. $\mathcal{O}$ is a functor from $\mathbf{Man}_n^{op}$ to **Set**.

G_n	$\mathrm{Obj}(G_n - \mathbf{Man}_n)$	$G_n - \mathrm{Str}$
$\mathrm{O}(n)$	$\mathrm{Obj}(\mathrm{Man}_n)$	$*$
$\mathrm{SO}(n)$	M with $w_1(M) = 0$	$\mathcal{O}$
$\mathrm{Pin}(n)$	M with $w_1(M) = w_2(M) = 0$	$H^1(\cdot, \mathbb{Z}_2)$
$\mathrm{Spin}(n)$	M with $w_1(M) = w_2(M) = 0$	$\mathcal{O} \times H^1(\cdot, \mathbb{Z}_2)$

Constructions emerging from geometric structures

Let $M \in \mathrm{Obj}(G_n - \mathbf{Man}_n)$.

- Each $g \in \mathrm{Met}(M)$ and $\alpha \in G_n - \mathrm{Str}(M)$ determines a G_n-principal bundle $PG_{g,\alpha}(M)$.
- The metric g induces a Levi-Civita connection 1-form ω^{LC} on $PG_{g,\alpha}(M)$.
- A G_n-representation $\rho : G_n \to \mathrm{End}(V)$ induces an associated vector bundle $E_{V,g,\alpha} = PG_{g,\alpha}(M) \times_\rho V$.
- The connection 1-form ω^{LC} induces a covariant derivative ∇^{LC} on $E_{V,g,\alpha}$.

Geometric symbol

Definition

Let V, W be hermitian representations of G_n. A geometric symbol/ universal elliptic symbol σ is a G_n-equivariant map

$$\sigma : \mathbb{R}^n \to \mathrm{Hom}(V, W),$$

such that for $\xi \in \mathbb{R}^n$, $\xi \neq 0$

$$\sigma(\xi) : V \to W$$

is an isomorphism.

Definition

For a Riemannian G_n-manifold (M, g, α), a geometric symbol σ defines an elliptic first-order differential operator

$$D_{\sigma,g,\alpha} := \bar{\sigma} \circ \nabla^{LC} : C^\infty(M, E_{V,g,\alpha}) \to C^\infty(M, E_{W,g,\alpha}),$$

where $\bar{\sigma}$ is the to σ associated section of $T^*M \otimes \mathrm{Hom}(E_{V,g,\alpha}, E_{W,g,\alpha})$. Operators constructed in this way are called geometric.

Definition of Chiral Geometric Symbol

Definition

Let $H_n \in \{\mathrm{Pin}(n), \mathrm{O}(n)\}$, V be an H_n-representation, and $G_n \subseteq H_n$ be the connected component of $1 \in H_n$. A chiral (G_n-)geometric symbol (σ, ε) consists of:

- A H_n-geometric symbol $\sigma : \mathbb{R}^n \to \mathrm{Hom}(V)$,
- a H_n-equivariant map $\varepsilon : \Lambda^n \mathbb{R}^n \to \mathrm{Hom}(V)$,

such that

- $\sigma(\xi)$ is skew-adjoint for all $\xi \in \mathbb{R}^n$
- $\varepsilon(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n)^2 = 1$
- $\sigma \cdot \varepsilon(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) = -\varepsilon(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) \cdot \sigma.$

Geometric Operators from Chiral Symbols

- Let $G_n \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$ and (σ, ε) be a chiral G_n-geometric symbol.
- $V^\pm$ are the ± 1-eigenspaces of $\varepsilon(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n)$.
- $V^\pm$ are G_n-representations and the restrictions

$$\sigma^\pm : \mathbb{R}^n \otimes V^\pm \to V^\mp$$

are G_n-equivariant.

Geometric Operators from Chiral Symbols

- On a Riemannian G_n-manifold (M, g, α), ε induces a $\mathbb{Z}_2$-grading:

$$E_V = E_{V^+} \oplus E_{V^-}$$

- $E_{V^\pm}$ are the ± 1-eigenspaces of $\bar{\varepsilon}(dvol_g)$.
- With respect to this splitting:

$$D_{\sigma,g,\alpha} = \begin{pmatrix} 0 & D_{\sigma^-,g,\alpha} \\ D_{\sigma^+,g,\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

- Since D_σ is self-adjoint:

$$(D_+)^* = D_-.$$

Index and heat kernel

W.r.to the splitting $E_V = E_{V^+} \oplus E_{V^-}$:

$$\exp(-tD^2_{\sigma,g,\alpha}) = \begin{pmatrix} \exp(-tD_-D_+) & 0 \\ 0 & \exp(-tD_+D_-) \end{pmatrix}.$$

$$\text{ind } D_+ = \int^{\mathcal{O}} \mathrm{str}(\exp(-tD^2_{\sigma,g,\alpha})(x,x))dvol_{g,\mathcal{O}},$$

where $\mathcal{O}$ is the orientation induced from α,

$$\mathrm{str}(A) = \mathrm{tr}(\bar{\varepsilon}(dvol_{g,\mathcal{O}})A), \quad A \in \mathrm{Hom}(E_V, E_V).$$

Atiyah-Singer index theorem

Denote by $\chi(TM_{\mathcal{O}}) \in H^n(M, \mathbb{R})$ the Euler class of the TM with respect to the orientation $\mathcal{O}$.

Theorem (Atiyah-Singer)

Let (σ, ε) be a chiral G_n-geometric symbol and (M, g, α) be a Riemannian G_n-manifold. Then

$$\mathrm{ind}(D_+) = (-1)^{\frac{n}{2}} \left(\frac{\mathrm{ch}(E_{+,g,\alpha}) - \mathrm{ch}(E_{-,g,\alpha})}{\chi(TM_{\mathcal{O}})} \cdot \hat{A}(M)^2 \right) [M_{\mathcal{O}}].$$

Atiyah-Singer integrand

Let

$$\left(\frac{\mathrm{ch}(E_{+,g,\alpha}) - \mathrm{ch}(E_{-,g,\alpha})}{\chi(TM_{\mathcal{O}})}(\nabla^{LC,g}) \cdot \hat{\mathrm{A}}(\nabla^{LC,g})\right)_n$$

be the n-form part of the Chern-Weyl form representing the characteristic class

$$\frac{\mathrm{ch}(E_{+,g,\alpha}) - \mathrm{ch}(E_{-,g,\alpha})}{\chi(TM_{\mathcal{O}})} \cdot \hat{A}(M)^2 \in H^*(M, \mathbb{R})$$

with respect to the Levi-Civita connections induced by g.

Using Gilkey's invariance theory one shows...

The local index theorem for geometric operators

Theorem

Let (σ, ε) be a chiral G_n-geometric symbol for n even. Let (M, g, α) be a Riemannian G_n-manifold and $D_{\sigma,g,\alpha} \colon C^\infty(M, E) \to C^\infty(M, E)$ be the induced geometric operator. Then the equality

$$\lim_{t \searrow 0} \left(\mathrm{str} \left(\exp\left(-tD^2_{\sigma,g,\alpha}\right)(x,x)\right) dvol_g \right) =$$
$$= (-1)^{\frac{n}{2}} \left(\frac{\mathrm{ch}(E_{+,g,\alpha}) - \mathrm{ch}(E_{-,g,\alpha})}{\chi(TM_{\mathcal{O}})} (\nabla^{LC,g}) \cdot \hat{\mathrm{A}}(\nabla^{LC,g})^2 \right)_n$$

holds.

Thank you for your attention!
Happy birthday to Prof. Bunke!