# Deformation Theory of $\mathbb{E}_\infty$-coalgebras

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# Motivation

Fix a prime $p$ and a space/anima $X$. Write $\mathbb{S}[X] := \Sigma^{\infty}_{+} X$ and $\mathbb{F}_p[X] := C_*(X; \mathbb{F}_p)$.

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**Question**

Can we recover $\mathbb{S}[X]^\wedge_p \in \mathrm{Sp}^\wedge_p$ from $\mathbb{F}_p[X] \in \mathrm{Mod}_{\mathbb{F}_p}$?

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Always have

$$\mathbb{S}[X]^{\wedge}_{p} \otimes \mathbb{F}_p \simeq \mathbb{F}_p[X]$$

but there no way to go back without more structure, e.g. $\mathbb{S}[\mathbb{R}P^2]^{\wedge}_{2}$ is not free over over $\mathbb{S}^{\wedge}_{2}$.

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## Refined Question

Given $X$, what is the space of $(A, \eta)$ where $A \in \mathrm{cCAlg}(\mathrm{Sp}_p^\wedge)$ and $\eta : A \otimes \mathbb{F}_p \simeq \mathbb{F}_p[X]$?

# Formally étale coalgebras

Let $R$ be an $\mathbb{E}_\infty$-ring. To any $R$-coalgebra $A$, we can associate a coalgebra

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**Proposition**

*Formally étale coalgebras admit unique lifts against square zero extensions $R' \to R$ i.e. we have*

$$\mathrm{Fib}_A\left(\mathrm{cCAlg}(\mathrm{Mod}_{R'}^{\mathrm{cn}}) \to \mathrm{cCAlg}(\mathrm{Mod}_R^{\mathrm{cn}})\right) \simeq *$$

**Theorem**

*For any (connected) space $X$ the coalgebra $R[X]$ is formally étale.*

# Result

The map $q : \mathbb{S}_p^{\wedge} \to \mathbb{F}_p$ induces an adjunction

$$q^* : \mathrm{cCAlg}(\mathrm{Sp}_p^{\wedge}) \leftrightarrows \mathrm{cCAlg}(\mathrm{Mod}_{\mathbb{F}_p}) : q_*$$

# Result

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**Theorem**

*The right adjoint $q_*$ induces a fully faithful functor*

$$\mathcal{W} : \mathrm{cCAlg}(\mathrm{Mod}_{\mathbb{F}_p}^{\mathrm{cn}})^{\mathrm{f\acute{e}t}} \to \mathrm{cCAlg}(\mathrm{Sp}_p^{\wedge,\mathrm{cn}})$$

*such that $\mathcal{W}(\mathbb{F}_p[X]) \simeq \mathbb{S}[X]_p^\wedge$ for any (connected) space $X$.*