Boundary value problems on manifolds with corners and conormal homology
Greifswald, May 2024

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From Analysis to Homotopy Theory
A conference in honor of Ulrich Bunke's 60th birthday

Ulrich Bunke's influence

Interactions: Search for, find and explore interactions between different areas of mathematics (and between mathematicians), a shared vision visible in much of Uli's work.

Oberwolfach Conferences "Analysis and Topology in Interaction"

Index theory on closed manifolds

Theorem (pre Atiyah-Singer)

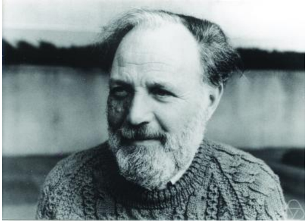
If M is a closed manifold and D an elliptic differential operator on M, then

$$D\colon H^1(M) \to L^2(M) \quad \text{is a Fredholm operator.}$$

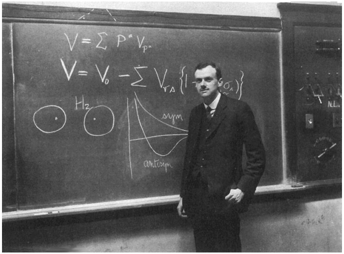
Theorem (Atiyah-Singer index theorem)

For the Dirac operator $\quad \operatorname{ind}(D) = \hat{A}(M)$.

Here $\hat{A}(M)$ is a differential-topological invariant.

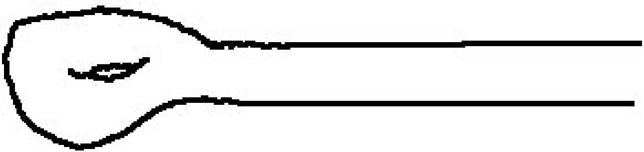


Index theory on manifolds with boundary



Dirac: Differential operator D as square root of matrix Laplacian (using Pauli matrices).

On compact manifolds with boundary, the Dirac operator is *not* a Fredholm operator with domain $H^1(M)$. One has to impose *boundary conditions*, defining the appropriate domain.



Boundary conditions for Dirac

- Classical Dirichlet or Neumann: no good in general!
- Atiyah-Patodi-Singer: use spectral boundary conditions!
- Melrose-Piazza, Bunke and others (during the study of *families* of boundary value problems): key is to *tame* the boundary operator, to get a Fredholm boundary problem
- **Definition** (Bunke). A taming is a smoothing perturbation making the resulting operator invertible
- **Atiyah-Patodi-Singer Index Theorem**. On a manifold with boundary, tamings always exist. The Fredholm index then is given by the Atiyah-Patodi-Singer index theorem

Excursion to differential K-theory

Differential cohomology (also known as Deligne cohomology, Simons-Sullivan differential characters) is a refinement of ordinary cohomology taking differential form information into account (a bigger group, not homotopy invariant).
Bunke's concept of tamings is used to construct a particularly rich model of differential K-theory (with all desirable structure), with cycles of geometric meaning and origin (families of Dirac type operators).

With Uli, we had a lot of fun discussing and writing several papers on this subject.

Boundary value problems on manifolds with corners

Definition

A smooth manifold with corners is a (generalized) smooth manifold, locally modelled on a "quadrant" in $[0,\infty)^k \times \mathbb{R}^{n-k}$.
It has faces of different codimension. The "being contained in" relations between the components of the faces is important combinatorial information about such a manifold.
Analysis: the operators are required to be translation invariant in all normal directions near the boundary faces.

Technicalities: these operators are extended from the "quadrant" to full flat space. L^2-index theory there translates to spectral boundary conditions.

Manifolds with corners

Definition

A compact smooth manifold with embedded corners is given by:

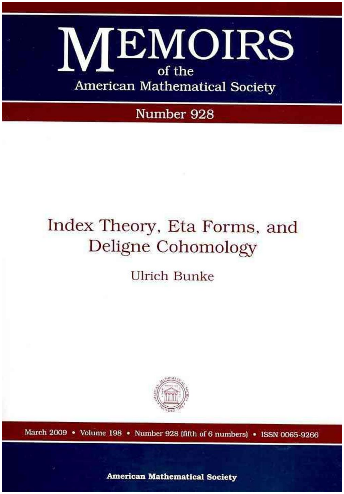
1. a closed smooth manifold $\tilde{X}$
2. boundary defining smooth maps $\rho_j\colon \tilde{X} \to \mathbb{R}$ defining

$$X := \bigcap \rho_j^{-1}([0,\infty)); \quad H_j := \rho_j^{-1}(0); \quad H_{j_1,\dots,j_k} := H_{j_1} \cap \dots \cap H_{j_k}$$

3. such that all faces are regular
4. each face $H_{j_1,\dots,j_k}$ of arbitrary codimension is connected

Admittedly: the connectedness condition is somewhat artificial and not required by all authors. But it is used in some of the analytic constructions implicit in later slides.

Fredholm boundary problems on manifolds with corners



"Given a Dirac operator on a manifold with boundary faces we use the tamings of its boundary reductions in order to turn the operator into a Fredholm operator. Its index is an obstruction against extending the taming from the boundary to the interior. In this way we develop an inductive procedure to associate Fredholm operators to Dirac operators on manifolds with corners and develop the associated obstruction theory."

The obstruction group

Theorem (Nazaikinskii, Savin, Sternin)

On X, a manifold with corners, the obstruction to taming is the boundary analytic index with values in the K-theory of b-compact operators on the boundary

$$K_*(\mathcal{K}_b(X)).$$

This K-theory depends only on the combinatorics of the faces of X.

Definition (Bunke)

The conormal homology $H_*^{cn}(X)$ is the homology of a chain complex generated abstractly by the faces of X.

Theorem (Carillo Rouse, Lescure, Velasquez)

There is a natural Chern character isomorphism

$$K_*(\mathcal{K}_b(X)) \otimes \mathbb{Q} \to H^{cn}_{*+2\mathbb{Z}}(X) \otimes \mathbb{Q}.$$

How nice is the obstruction group?

Observation

In low codimension, the obstruction group has nice properties:

1. *For manifolds just with boundary and of even dimension, it vanishes.*
2. *If all faces are of codimension ≤ 3, it is torsion free and the Chern character is an integral isomorphism.*

Question (Carillo Rouse, Lescure, Velasquez)

Is the conormal homology always torsion free, and the Chern character an integral isomorphism?

Constructions

Theorem (S., Velasquez)

Given a manifold with embedded corners X, we construct the simplicial complex Σ_X encoding the corner structure. It satisfies:

$$K_{-*-1}(\Sigma_X) \cong K_*(\mathcal{K}_b(X)); \qquad H_{*-1}(\Sigma_X) \cong H_*^{cn}(X).$$

Theorem (S., Velasquez)

For every finite simplicial complex C there is a smooth manifold with embedded corners X such that

$$\Sigma_X \cong C.$$

Corollary

The answer to the above questions is a drastic "NO".

Corner structure complex

Definition

Let X be a manifold with embedded corners with $n+1$ boundary faces $H_0, \ldots, H_n$. Its *corner structure complex* is the abstract simplicial complex with vertex set $\{H_0, \ldots, H_n\}$ and for $A \subset \{H_0, \ldots, H_n\}$ we set

$$A \in \Sigma_X \iff \bigcap_{H_i \in A} H_i \neq \emptyset$$

Example: for the cube, the corner structure complex is the (boundary of) the octahedron.

Corner complex constructions

Lemma

If X_1, X_2 are manifolds with embedded corners, then

$$\Sigma_{X_1 \times X_1} \cong \Sigma_{X_1} * \Sigma_{X_2}$$

Recall that the join of two abstract simplicial complexes has vertex set the disjoint union of vertex set, and simplices the unions of simplices.

Example

1. The product of n manifolds with non-empty connected boundary is a manifold with embedded corners X_n and with $\Sigma_{X_n} = \Delta_n$, the standard n-simplex.
2. Directly, there is an explicit $n+2$-dimensional A with $\Sigma_A = \Delta_n$: $A = S^1 \times Q^{n+1} \cup_{S^1 \times (S^n \cap Q^{n+1})} D^2 \times (S^n \cap Q^{n+1}) \subset S^{n+2} = \partial(D^2 \times D^{n+1})$

Connected sum

Definition

Given two manifolds with corners X and Y with points $x_0 \in X$, $y_0 \in Y$ with an isomorphism of local corner structures near x_0 and y_0.
Then one can form the connected sum along x_0 and y_0 (gluing the corresponding faces near the points), a manifold with corners (but not always with connected faces), and $\Sigma_{X\#Y} = \Sigma_X \cup_{\Delta_k} \Sigma_Y$.

Lemma

Given a compact manifold with embedded corners X and an embedding $\iota\colon \partial\Delta_n \hookrightarrow \Sigma_X$ which can not be filled.
Then, doing an n-fold connected sum with the model A_n one obtains a manifold with embedded corners Z and $\Sigma_Z = \Sigma_X \cup_\iota \Delta_n$.

Corollary

Inductively, this produces X with Σ_X any given C.

Orbit space O_X

Given a manifold with embedded corners X, there is an associated free and proper groupoid (the puff groupoid) with orbit space O_X (a non-compact amplification of X for large m) and with a Connes-Thom isomorphism

$$CT\colon K_*(\mathcal{K}_b(X)) \xrightarrow{\cong} K^{m+*}(O_X).$$

By a direct computation, its one-point compactification satisfies

$$O_X^+ \cong S^{m-|V|+1} \wedge (|\Delta(V)|/|\Sigma_X^\vee|).$$

Here V is the vertex set of Σ_X and $\Sigma_X^\vee$ the dual complex inside the simplex spanned by V. Then

$$\begin{aligned} K^{m+*}(O_X) &\overset{\text{Def}}{=} \tilde{K}^{m+*}(O_X^+) \\ &\overset{\text{susp.isp}}{\cong} \tilde{K}^{*+|V|-1}(|\Delta(V)|/|\Sigma_X^\vee|) \\ &\overset{\text{pair boundary}}{\cong} \tilde{K}^{*+|V|-2}(|\Sigma_X^\vee|) \\ &\overset{\text{Whitehead duality}}{\cong} \tilde{K}_{-*-1}(|\Sigma_X|) \end{aligned}$$

Summary

For conormal homology: identify the chain complexes!

Lesson learned

- sophisticated analysis questions sometimes can be answered using elementary homology (conormal chain complex).
- If a question sounds to good to be true: look for a counterexample!
- Use systematic constructions!

Problem

- *Overcome the annoying connectedness assumption!*
- *How about "higher indices" (coefficients in C^*-algebra bundles)?*
- *can we learn more from/give more to differential K-theory?*

Thank you!

THANK YOU for your attention (to all)!

THANK YOU for the opportunity to give this presentation (to organizers)!

Lieber Uli: VIELEN DANK. . . !

and a belated HAPPY BIRTHDAY!