Quantum observables for free fermions with boundary conditions — Greifswald, May 17, 2024

joint with Bill Dwyer

Thm (Lichnerowicz 1963):

X closed Riemannian spin mfd with $s(x) > 0$ for all $x \in X$. (scalar curvature)

Then $0 = \hat{A}(X) \in \mathbb{R}$ — Hirzebruch's $\hat{A}$-genus of X

Rem: i) X admits a spin structure $\Leftrightarrow$ $TX_{|X^{(2)}}$ trivial $\Leftrightarrow$ X orientable and $0 = w_2(X) \in H^2(X;\mathbb{Z}/2)$

ii) spin assumption essential since $\hat{A}(\mathbb{C}P^2) = -\frac{1}{8}$, $s > 0$ for the standard metric

iii) $\hat{A}(X) = \operatorname{index}(D_X)$ — Atiyah-Singer Index Theorem; D_X: Dirac operator on X

Conjecture (Hoehn, S. 1996) X closed Riem. string mfd with $\mathrm{Ric}_x > 0$ for all $x \in X$. (Ricci curvature)

Then $0 = \mathrm{Wit}(X) = W_0(X) + W_1(X)q + W_2(X)q^2 + \ldots \in \mathbb{C}[[q]]$, with $W_0(X) = \hat{A}(X)$

Witten-genus of X

Rem: i) X admits a string structure $\Leftrightarrow$ $TX_{|X^{(4)}}$ is trivial $\Leftrightarrow$ X spin & $0 = \lambda(X) \in H^4(X;\mathbb{Z})$

$2\lambda(X) = p_1(X)$

ii) string assumption essential: $\mathrm{Wit}(\mathbb{H}P^2) \neq 0$, but $\mathrm{Ric} > 0$

Rem: Witten offered two interpretations of $Wit(X)$:

ⓐ $Wit(X) =$ S^1-equivariant index of the Dirac operator D_{LX} on $LX = \{\gamma : S^1 \to X\}$ free loop space of X.

ⓑ $Wit(X)$ is the partition function of a 2-dimensional QFT σ_X, the super symmetric non-linear σ-model of X, i.e. for $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$

$$Wit(X)(q) = \sigma_X(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau) \in K_\tau^{\otimes n} ; \quad n = \dim X, \quad K \to \mathbb{H} \text{ holomorphic line bdl, } SL_2(\mathbb{Z})\text{-equivariant}$$

Results concerning D_{LX} resp. σ_X that might be useful for proving the conjecture.

ThmA (Taubes 1989): $\exists$ S^1-equivariant operator D_{LX} with $\mathrm{index}^{S^1}(D_{LX}) = Wit(X)$.

Problem: String condition not needed for construction!

ThmB (Costello 2010, Gorbounov-Gwilliam-Williams 2020):

$\exists$ 2-dim QFT σ_X with partition function $Wit(X)$ for complex mfd X with $0 = p_1(X) \in H^4(X;\mathbb{Q})$.

Problem: Construction requires a complex structure on X, but no Riem. metric.

ThmC (Bertram Arnold's thesis, 2022) Same as above, but with Riem. metric instead of cx. structure

Suspected problem: the assumptions spin & $0 = \lambda(X) \in H^4(X;\mathbb{Z})$ are needed for conjecture.

Tool used to construct the QFT G_X of Thm B & C:

Input

a classical field theory consisting of:
- a spacetime manifold M
- a space of fields $\mathcal{F}_M$
- an action functional $S: \mathcal{F}_M \to \mathbb{R}$

$\longrightarrow$ perturbative quantization procedure of Costello-Gwilliam $\longrightarrow$

$\updownarrow$

homological approach to quantization described in the books:
Renormalization and effective Field Theory, Costello 2011
Factorization algebras in QFTs, Costello-Gwilliam, Vol. I (2017), Vol II (2021)

Output

Factorization algebra Obs^q of quantum observables on M, i.e.
functor $Obs: Open(M) \longrightarrow Ch$

objects: $U \underset{open}{\subset} M$ cochain cx.

morphisms: $U \hookrightarrow U'$ cochain maps

with properties
- multiplicativity: U, V disjoint
  $Obs^q(U) \otimes Obs^q(V) \xrightarrow[w.e.]{\sim} Obs^q(U \sqcup V)$
- cosheaf property (expressing locality of quantum observables)

Ex: particle moving in a Riemannian mfd X

- $M = \mathbb{R}$
- $\mathcal{F}_M = C^\infty(\mathbb{R}, X)$
- $S: C^\infty(\mathbb{R}, X) \to \mathbb{R}$
  $S(\phi) := \int_{\mathbb{R}} |d_x\phi|^2 dx$

classical fields $= \{$critical points of $S\} = \{$geodesics in $X\} \leftrightarrow TX$

$\gamma \longmapsto (\gamma(0), \gamma'(0))$

algebra of classical observables $:= C^\infty($classical fields$) = C^\infty(TX)$

algebra of quantum observables $= D_X =$ algebra of diff. ops on X

acts on the vector space $C^\infty(X)$ (of quantum states)

Ex: the 1D free boson, aka particle moving in a real inner product space V

$M=\mathbb{R}$, $\mathcal{U} \underset{\text{open}}{\subset} M$, $\mathcal{F}_M(\mathcal{U}) = C^\infty(\mathcal{U},V)$, $S(\phi) = \int_{\mathcal{U}} |\phi'(x)|^2 dx = \int_{\mathcal{U}} \langle \phi, \Delta\phi \rangle dx$

$\mathrm{Crit}(S) = \left\{ \begin{array}{l} \text{geodesics in } V \\ \text{parametrized by } \mathcal{U} \end{array} \right\} = \mathrm{Ker}\left(C^\infty(\mathcal{U},V) \xrightarrow{\Delta} C^\infty(\mathcal{U},V) \right) \cong V \oplus V$ for $\mathcal{U}=(a,b)$

$\sim \downarrow$ q.i. $\qquad (t \mapsto q+tp) \leftarrow (q,p)$

cochain complex of BV-fields $\mathcal{E}(\mathcal{U}) := \left(C^\infty(\mathcal{U},V) \xrightarrow{\Delta} C^\infty(\mathcal{U},V) \right)$

deg $\quad 0 \qquad\qquad 1$

Facts:

1) W vectorspace, $W^\vee := \mathrm{Hom}(W,\mathbb{R}) = \{\text{linear functions on } W\}$

$\{\text{polynomial functions on } W\} := \mathrm{Sym}^*(W^\vee) = \bigoplus_k \mathrm{Sym}^k(W^\vee)$

2) $\mathcal{E}^\vee :=$ dual to $\mathcal{E}$

deg $\quad -1 \qquad\qquad 0$

$\mathcal{E}^\vee(\mathcal{U}) = \left(C^\infty(\mathcal{U},V)^\vee \xrightarrow{\Delta^\vee} C^\infty(\mathcal{U},V)^\vee \right)$

$\sim \uparrow$ q.i. $\qquad \uparrow \qquad\qquad \uparrow$

$\mathcal{E}_c(\mathcal{U})[1] := \left(C_c^\infty(\mathcal{U},V) \xrightarrow{\Delta} C_c^\infty(\mathcal{U},V) \right)$

motivates

> **Definition:**
>
> $\mathrm{Obs}^{cl}(\mathcal{U}) := \mathrm{Sym}^*(\mathcal{E}_c(\mathcal{U})[1])$
>
> cochain complex of classical observables
>
> gives factorization algebra Obs^{cl} on $\mathbb{R}$.

note: $H^0\mathrm{Obs}^{cl}((a,b)) = \mathrm{Sym}^*(H^0\mathcal{E}_c(\mathcal{U})[1]) = \mathrm{Sym}^*((V\oplus V)^\vee) = \mathbb{C}[\underbrace{q_1,\dots,q_n}_{\text{positions}}, \underbrace{p_1,\dots,p_n}_{\text{momenta}}] \subset C^\infty(TV)$

$$\mathcal{E}_c(U)[1] = \Big(\underset{\substack{\rotatebox{90}{$\in$} \\ \phi}}{\overset{-1}{C_c^\infty(U,V)}} \xrightarrow{\Delta} \underset{\substack{\rotatebox{90}{$\in$} \\ \psi}}{\overset{0}{C_c^\infty(U,V)}} \Big) \qquad \{\phi,\psi\} := \int_U \phi\psi \, dx \quad \text{pairing of degree } +1$$

extend to a pairing $\{\ ,\ \}$ of deg $+1$ on $\mathrm{Sym}^*(\mathcal{E}_c(U)[1])$ which is a derivation in both slots

$\hookleftarrow$ degree $+1$ Poisson bracket on $\mathcal{E}_c(U)[1]$

determines a unique homomorphism of degree $+1$

$\Delta_{BV} : \mathrm{Sym}^k(\mathcal{E}_c(U)[1]) \longrightarrow \mathrm{Sym}^{k-2}(\mathcal{E}_c(U)[1])$ called BV-Laplacian char. by

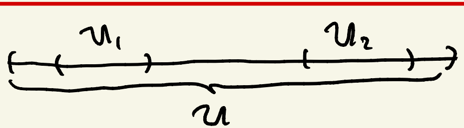
$$\Delta_{BV}(ab) = \Delta_{BV}(a)\,b \pm a\,\Delta_{BV}(b) + \{a,b\}$$

$Q :=$ differential on $\mathrm{Sym}^*(\mathcal{E}_c(U)[1])$ induced by differential on $\mathcal{E}_c(U)[1]$

Fact: $Q + \Delta_{BV}$ is again a differential, i.e. $(Q+\Delta_{BV})^2 = 0$

$$\boxed{\text{Def: } \mathrm{Obs}^q(U) := \big(\mathrm{Sym}^*(\mathcal{E}_c(U)[1]),\ Q+\Delta_{BV}\big)}$$

⚠ the algebra structure on $\mathrm{Obs}^q(U)$ does NOT induce an algebra structure on $H^0\mathrm{Obs}^q(U)$, since Δ_{BV} is not a derivation.



$\Rightarrow \mathrm{Obs}^q(U_1) \otimes \mathrm{Obs}^q(U_2) \to \mathrm{Obs}^q(U)$ chain map

$\Rightarrow H^0\mathrm{Obs}^q(U_1) \otimes H^0\mathrm{Obs}^q(U_2) \longrightarrow H^0\mathrm{Obs}^q(U) \qquad (*)$

Facts:

i) Obs^q is a locally constant factorization algebra on $\mathbb{R}$, i.e.

$(a,b) \hookrightarrow (c,d) \Rightarrow H^0\mathrm{Obs}^q((a,b)) \xrightarrow{\simeq} H^0\mathrm{Obs}^q((c,d))$

in particular, the map $(*)$ given by "stacking of intervals" gives $H^0\mathrm{Obs}^q(\mathbb{R})$ an algebra structure.

ii) $H^0\mathrm{Obs}^q(\mathbb{R}) \cong$ Weyl algebra with generators $q_1,\dots,q_n,p_1,\dots,p_n$ and relations $[p_i,q_j]=\delta_{ij}$ $\cong$ differential operators with polynomial coefficients on V

(iii) Obs^q can be extended to a bulk-boundary factorization algebra by defining $\mathrm{Obs}^q([a,b))$ as above but using Neumann boundary conditions at a for the complex $\mathcal{E}$.

iv) $H^0\mathrm{Obs}^q([a,b)) \cong \mathrm{Sym}^*(V^\vee) = \mathbb{C}[q_1,\dots,q_n]$ is a module over $H^0\mathrm{Obs}^q(\mathbb{R})$ via stacking. This gives the standard structure of $\mathbb{C}[q_1,\dots,q_n]$ as module over the Weyl algebra

Variation: the 1D free fermion

$M = \mathbb{R} \underset{\text{open}}{\supset} U$

$$\begin{array}{llll} \text{deg} & 0 & & 1 \\ \mathcal{E}(U) := (& \underset{\substack{\uparrow \\ \text{parity reversal}}}{\Pi} C^\infty(U,V) & \xrightarrow{d} & \Pi\Omega^1(U,V)) \end{array}$$

chain cx. of fields

$$\begin{array}{llll} & -1 & & 0 \\ \mathcal{E}_c(U)[1] := (& \Pi C^\infty_c(U,V) & \xrightarrow{d} & \Pi\Omega^1_c(U,V)) \end{array}$$

chain cx. of linear observables

equipped with pairing $\{,\}$ of deg $+1$

$$\mathrm{Obs}^{cl}(U) := \mathrm{Sym}^*(\mathcal{E}_c(U)[1])$$

$$\mathrm{Obs}^q(U) := (\mathrm{Sym}^*(\mathcal{E}_c(U)[1]), Q+\Delta_{BV})$$

$$H^0\,\mathrm{Obs}^{cl}((a,b)) = \mathrm{Sym}^*(\Pi V^\vee) = \underbrace{\Lambda^*(V^\vee)}_{\mathbb{Z}/2\text{-graded alg}}$$

fact: $H^0\mathrm{Obs}^q((a,b)) \cong Cl(V) = T(V)/_{v\cdot v = \langle v,v\rangle\cdot 1}$

algebra structure given by stacking of intervals

algebra isom.

Clifford algebra

boundary conditions?

need subspaces $L \subset V$ which are Lagrangian, i.e. $L^\perp = L$. There are none since bilinear form on V is positive definite.

Trick: replace V by $W := \underbrace{-\mathbb{R}^n}_{\text{neg. def.}} \oplus \underbrace{V}_{\text{pos definite}}$

For a Lagrangian $L \subset W$ define

$\mathcal{E}_L([a,b))$ by using boundary condition $\phi(a) \in L$ for $\phi \in C^\infty([a,b),V)$.

$\mathrm{Obs}^q_L([a,b)) := \left(\mathrm{Sym}^*(\mathcal{E}_{L,c}([a,b))[1]), Q + \Delta_{BV}\right)$

Prop: $H^0 \mathrm{Obs}^q_L([a,b)) \cong \mathcal{Cl}(W) \otimes_{\mathcal{Cl}(L)} \mathbb{R} =: F_L$ Fockspace

module structure given by stacking

as $\mathcal{Cl}(W)$-modules

$\mathcal{Cl}(L) \cong \Lambda^* L$ since bilinear form vanishes on L

fiber bundle F with fiber F_L over $L \in \mathrm{Lag}(W)$

$$\begin{array}{c} F \\ \downarrow \\ \{L \subset W / L^\perp = L\} =: \mathrm{Lag}(W) \cong O(\mathbb{R}^n; V) := \{\mathbb{R}^n \xrightarrow{\text{isometry}} V\} \end{array}$$

has structure group $\mathbb{Z}/2$ (since F_L is an irreducible $\mathcal{Cl}(W)$-module)

$\Rightarrow \exists$ flat connection, holonomy on $\mathrm{Lag}^+(W) \cong SO(\mathbb{R}^n, V)$ is nontrivial.

Pulling back to a $\underbrace{\text{non-trivial double covering } \widetilde{SO}(\mathbb{R}^n, V) \to SO(\mathbb{R}^n, V)}_{\text{spin structure on } V}$,

allows a natural identification of

$H^0 \mathrm{Obs}^q_L([a,b))$ and $H^0 \mathrm{Obs}^q_{L'}([a,b))$ for $L, L' \in \mathrm{Lag}(W)$.

Upshot: a spin structure on V allows to extend the 1D free fermion to a bulk-boundary theory.

Conjecture: A string structure on V allows to extend the 2D free fermion to a bulk-boundary theory.