From analysis to homotopy theory
Greifswald 2024,
celebrating Uli.

Immersion order on 4-manifolds

Joint paper with
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10 years ago in Greifswald ...
Uli was as fast in the sand as in lectures.

The immersion order on 4-manifolds

Motivation from quantum cellular automata:

- a cellular automaton is a local rule for updating states at some collection of sites, often points on a lattice or a grid in a manifold, e.g. Conway's game life.
- a QCA is a local unitary evolution, acting on a collection of Hilbert-spaces.

$$H_1 \cdot \; \cdot H_2 \xrightarrow{\text{collide}} \cdot \; H_1 \otimes H_2$$

- $M \longmapsto QCA(M)$ leads to a coarse (!?) homology theory that has (contravariant) transfer maps for codim. 0 immersions $M \hookrightarrow N$ and in fact for $M \setminus \{m_0\} =: M_* \hookrightarrow N$ healing the puncture if M^d, N^d are closed, connected.

Def.: 1) $M \leq N$ if $\exists\, M_* \hookrightarrow N$

2) $M \bowtie N$ if $M \leq N$ & $N \leq M$

immersion equivalence classes inherit partial order

S^n is the minimum, $M_1, M_2 \leq N \Rightarrow M_1 \# M_2 \leq N$

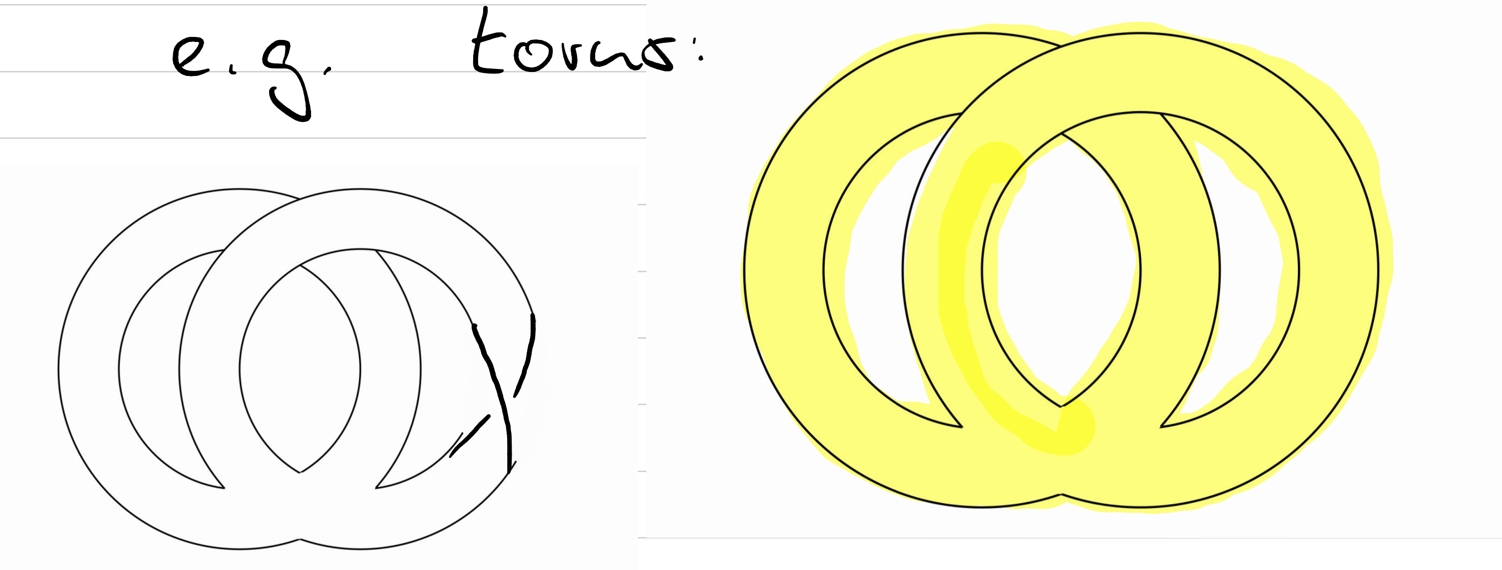
Mike's question: For $d=4$, which orders appear?

Warm-up : $d = 2$. Then $S^2 < \mathbb{R}P^2$ represents the entire immersion order (on closed, connected 2-manifolds)

Proof : $S^d \leq M^d$ is absolute minimum $\forall d$

If M^2 is orientable then $M_* \hookrightarrow \mathbb{R}^2 \subseteq S^2$

e.g. torus:



and similarly for higher genus (and non-orient.)

$j : M_* \hookrightarrow N \Rightarrow j_* w_i(TN) = w_i(TM)$ so $S^2 \not< \mathbb{R}P^2$

Prop. $d=3$: $M^3 \leq N^3 \iff \exists$ homom. $\varphi : \pi_1 M \to \pi_1 N$

$$\begin{tikzcd} \pi_1 M \arrow[rr, "\varphi"] \arrow[dr, "w_1"'] & & \pi_1 N \arrow[dl, "w_1"] \\ & \mathbb{Z}/2 & \end{tikzcd}$$

$w_2 M := w_2 \tau M = 0$

1) $S^3 \bowtie M \iff M$ orientable

2) $\mathbb{RP}^2 \times S^1$ is global maximum and $\mathbb{RP}^2 \times S^1 \bowtie N$

$\iff \exists\, x \in \pi_1 N : x^2 = 1$ but $w_1(x) \neq 0$.

3) $S^1 \tilde{\times} S^2 \leq N \iff N$ is non-orientable

4) $M \leq S^1 \tilde{\times} S^2 \iff \exists$

$$\begin{tikzcd} \pi_1 M \arrow[r, "w_1"] \arrow[dr, dashed] & \mathbb{Z}/2 \\ & \mathbb{Z} \arrow[u, two heads] \end{tikzcd}$$

Alan Reid's appendix is about a hyperbolic M^3 s.t. $S^1 \tilde{\times} S^2 \lneqq M \lneqq \mathbb{RP}^2 \times S^1$.

Question: Does this immersion order contain ∞ chains $\cdots \lneqq M_0 \lneqq M_1 \lneqq M_2 \lneqq \cdots$? $d=4$ ✓

Prop.:
$d=4$

1) $M^4 \leq S^4 \iff M$ spin, i.e. $w_1 M = w_2 M = 0$

2) $M \leq \mathbb{C}P^2 \iff M$ orientable (∃ max.?)

3) $\mathbb{C}P^2 \leq M \iff w_2 \tilde{M} \neq 0$

4) $S^1 \tilde{\times} S^3 \leq M \iff M$ non-orientable

5) $M \leq S^1 \tilde{\times} S^3 \iff w_1 M$ lifts to $\mathbb{Z}$, $w_2 M = 0$

Proofs: By h-principle (or Phillips' thm.)

$j: M_* \hookrightarrow N$ exists $\iff$
$$\begin{array}{ccc} TM_* & \hookrightarrow & TN \\ \downarrow & & \downarrow \\ M_* & \xrightarrow{f} & N \end{array}$$
exists

Moreover, 4-dim. vectorbundles over 3-complexes are classified by w_i and the Wu-formulas give $w_3 M = Sq^1 w_2 M + w_1 M \cup w_2 M$

Prop.: $\exists\, M^4_* \hookrightarrow N^4 \iff \exists\, M_* \xrightarrow{f} N$ with $f_* w_i N = w_i M$, $i=1,2$.

$$\iff \exists \quad \begin{tikzcd} \pi_1 M \arrow[rr] \arrow[dr, "w_1"'] & & \pi_1 N \arrow[dl, "w_1"] \\ & \mathbb{Z}/2 & \end{tikzcd} \quad \& \quad \begin{tikzcd} C_2M/\operatorname{im} d_3^M \arrow[rr, "h"] \arrow[dr, "w_2"'] & & C_2N/\operatorname{im} d_3^N \arrow[dl, "w_2"] \\ & \mathbb{Z}/2 & \end{tikzcd} \quad \text{s.t. } d_2^N \circ h = C_1(\varphi)\circ d_2^M$$

From analysis to homotopy theory

Cor: The quadruple $(\pi_1 M,\ w_1 M,\ w_2^\pi M;\ \pm c_*[M])$ determines immersion equiv. class!

$w_2^\pi M \in H^2(\pi_1 M; \mathbb{Z}/2) \cup \{\infty\}$; $\quad c_*[M] \in H_4(\pi_1 M; \mathbb{Z}^{w_1})$

e.g. immersion order on 4-mfds with $\pi_1 \cong \mathbb{Z}$:

$$\begin{tikzcd} & & S^1\times S^3 \# \mathbb{CP}^2 \arrow[dr] & \\ S^4 \bowtie & S^1 \times S^3 \arrow[ur] \arrow[dr] & & S^1 \tilde{\times} S^3 \# \mathbb{CP}^2 \\ & & S^1 \tilde{\times} S^3 \arrow[ur] & \end{tikzcd}$$

Order graph

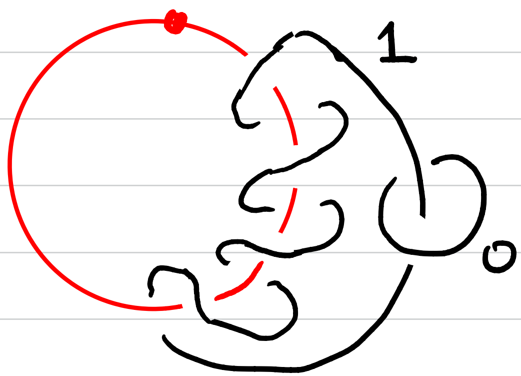
Theorem 1: Immersion equivalence classes of orientable 4-manifolds with cyclic fundamental group form the ∞ chain

$$S^4 < M(2) < M(4) < \ldots < M(2^m) < \ldots < \mathbb{CP}^2$$

In terms of order graphs this look as follows.

$$S^4 \longrightarrow M(2) \longrightarrow M(4) \longrightarrow \ldots \longrightarrow M(2^m) \longrightarrow \ldots \longrightarrow \mathbb{CP}^2$$

e.g. $M(4) =$



$:= \underbrace{h^0 \cup h^1}_{D^4 \setminus \nu D^2} \cup 2h^2 \cup \underbrace{h^3 \cup h^4}_{\exists!\; S^1 \times D^3}$

$M(2^m)$ is not spin but almost spin, $\pi_1 \cong \mathbb{Z}/2^m$

$\approx$

$M(2^m \cdot k)$ $\forall$ odd k, rational homology 4-spheres

$$\begin{tikzcd} S(3\cdot\gamma) \arrow[r] \arrow[d, equal] & M(2) \arrow[r, "\pi"] & \mathbb{R}P^2 \\ S^2 \arrow[r] & \widetilde{M}(2) \arrow[r] \arrow[u] & S^2 \arrow[u] \end{tikzcd}$$

$$w(TM(2)) = w\,\pi^*(3\gamma \oplus T\mathbb{R}P^2)$$
$$= \pi^*(1+a)^6 = \pi^*(1+a^2)$$
$$0 \neq a \in H^1(\mathbb{R}P^2; \mathbb{Z}/2).$$

$$\widetilde{M}(2) \cong S^2 \times S^2, \quad \tau(x,y) = (-x,-y) \text{ free action}$$

$\tau = u^2$, $u(x,y) := (y, -x)$ also free

$u^4 = \tau^2 = \mathrm{id}$ so $S^2 \times S^2 / u$ has $\pi_1 \cong \mathbb{Z}/4$

Is a (non-orientable) !! and $\chi = 1$

rational homology 4-balls: $N(4,1,1)$ and also

$N(2,1,1) := \mathbb{R}P^4$

Theorem 2: The immersion order graph for non-orientable 4-mfds. with cyclic π_1 is

$$\begin{tikzcd}
S^1\widetilde{\times}S^3 \arrow[r] \arrow[d] & \cdots \arrow[rr] \arrow[ddr] & & N(2^{m+1},0,0) \arrow[rrr] \arrow[d] \arrow[ddrr] & & & N(2^m,0,0) \arrow[rr] \arrow[d] \arrow[ddrr] & & \cdots \\
S^1\widetilde{\times}S^3\#\mathbb{CP}^2 \arrow[r] & \cdots \arrow[rr] & & N(2^{m+1},\infty,0) \arrow[rrr] \arrow[dr] & & & N(2^m,\infty,0) \arrow[rr] \arrow[dr] & & \cdots \\
\cdots & \cdots & N(2^{m+1},1,0) \arrow[ur] \arrow[dr] & & N(2^{m+1},\infty,1) & N(2^m,1,0) \arrow[ur] \arrow[dr] & & N(2^m,\infty,1) & \cdots \\
& \cdots & & N(2^{m+1},1,1) \arrow[ur] & & & N(2^m,1,1) \arrow[ur] & & \cdots
\end{tikzcd}$$

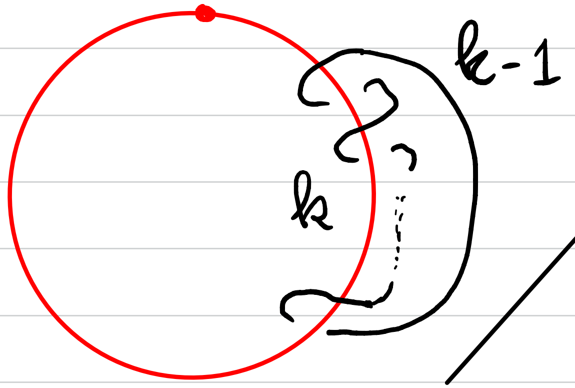
$N(2^m, w_2, c)$ has $\pi_1 \cong \mathbb{Z}/2^m$, $w_2 \in H^2(\pi_1;\mathbb{Z}/2) \cup \{\infty\}$

$$\overset{\text{⧗}}{} \qquad c \in H_4(\pi_1;\mathbb{Z}^-) \cong \mathbb{Z}/2 \qquad H^2(\pi_1;\mathbb{Z}/2) = \{0,1\}$$

$N(2^m \cdot k, w_2, c)$ $\forall$ odd k.

Constructions: $N(2k, w_2, 0)$ as before!

$N(2k, \infty, c) := N(2k, 1, c) \# \mathbb{C}P^2$

$N(2k, 1, 1) :=$  $/$ free involution on $\partial \cong L(k^2, k-1)$

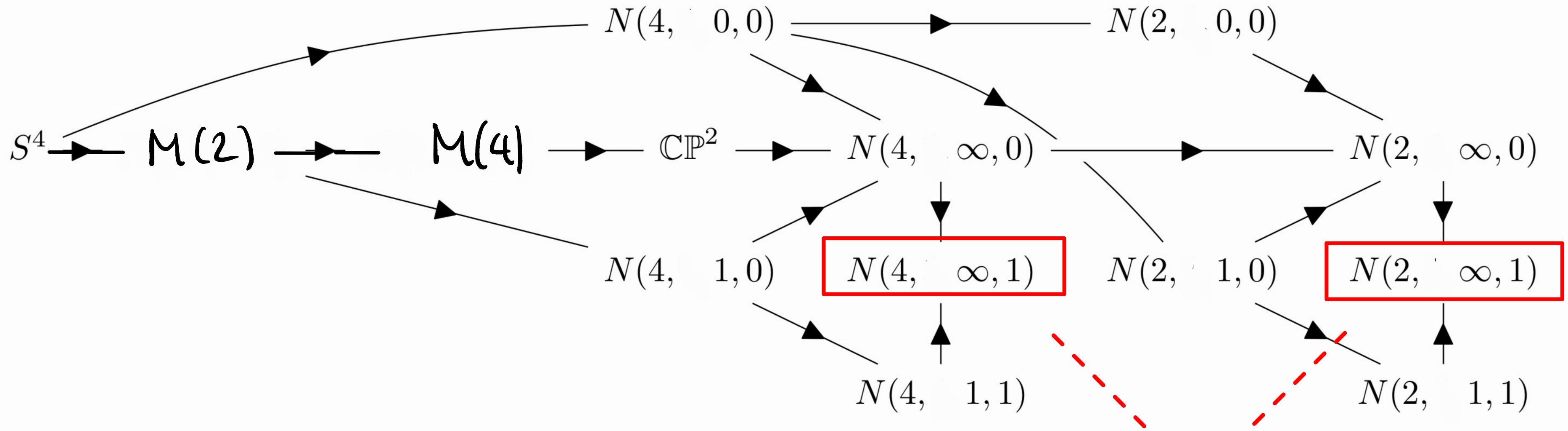
e.g. $k = 1$: $N(2, 1, 1) = D^4 /$ free involution on $\partial \cong S^3$

$\cong \mathbb{R}P^4$.

Prop.: $M(2^q) \le N(2^m, w_2, c) \iff w_2 = \infty$ or $\begin{cases} w_2 = 1 \\ q < m \end{cases}$

Get $\le$ on all 4-mfds with cyclic π_1.

e.g. $\pi_1 \in \{0, \mathbb{Z}/2, \mathbb{Z}/4\}$ gives partial order graph



local maxima