High-dimensional expansion and soficity of groups

Andreas Thom From Analysis to Homotopy Theory Mai 2024, Greifswald







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Overview

- 1. Equations over groups
- 2. Finitary approximation properties of groups
- 3. High-dimensional expansion and soficity of groups

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- Embed Q ⊂ C, study the continuous map p: C → C, and use a topological argument to see that there exists α ∈ C, such that p(α) = 0.

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Equations over groups - the one variable case

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Definition Let Γ be a group and let $g_1, \ldots, g_n \in \Gamma$, $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Z}$.

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The equation has a solution **over** Γ if there is an extension $\Gamma \leq \Lambda$ and there is some $h \in \Lambda$ such that w(h) = 1 in Λ .

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The study of equations like this goes back to:

Bernhard H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc. 18 (1943), 4-11.

Each equation

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- When is w trivial, i.e., only 1 is in the image?
- When is w surjective?
- Is every self-map of Γ of this form?

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The equation $w(t) = tat^{-1}ata^{-1}t^{-1}a^{-2}$ cannot be solved over $\mathbb{Z}/p\mathbb{Z} = \langle a \rangle$.

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Indeed, if w(t) = 1, then

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and a conjugate of *a* (namely tat^{-1}) would conjugate *a* to a^2 . But the automorphism of $\mathbb{Z}/p\mathbb{Z}$ which sends 1 to 2 has order dividing p-1 and hence the order is co-prime to *p*.

We say that the equation $w(t) = g_1 t^{\varepsilon_1} g_2 t^{\varepsilon_2} g_3 \dots g_n t^{\varepsilon_n}$ is non-singular if $\sum_{i=1}^n \varepsilon_i \neq 0$.

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If Γ is torsionfree and w(t) is non-singular, then w(t) can be solved over Γ .

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We will focus on the second conjecture.

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The resulting effect on fundamental groups is exactly

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Question (Connes, 1978)

Do all groups have this approximation property?

Any non-singular equation with coefficients in a finite group Γ can be solved over Γ . In fact, a finite extension is enough.

The same argument works for any group that is residually finite or, more generally, suitably approximated by unitary groups.

Such groups are called Connes-embeddable, more precisely,

$$\Gamma \subset \left(\prod_{n} \mathrm{U}(n)\right) / N.$$

Corollary (Pestov, 2009)

Any Connes-embeddable group Γ satisfies Kervaire's Conjecture.

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Equations over groups – many equations and variables

Let $w_1, w_2, \ldots, w_k \in \mathbb{F}_n * G$, and denote by $\varepsilon(w_1), \ldots, \varepsilon(w_k) \in \mathbb{F}_n$ their images under the natural homomorphism $\varepsilon \colon \mathbb{F}_n * G \to \mathbb{F}_n$.

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Under what conditions on $\varepsilon(w_1), \ldots, \varepsilon(w_k)$ is the homomorphism

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Question

Consider $X \subset Y \to Y/X$, with Y/X two-dimensional. When is $\pi_1(X) \to \pi_1(Y)$ injective?

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It is possible to solve one equation $w \in \mathbb{F}_n * G$, when $\varepsilon(w) \neq 1$.

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Let G be a Connes-embeddable group and let $w_1, \ldots, w_k \in \mathbb{F}_n * G$. If the presentation complex of the presentation

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admits a covering with trivial second homology, then the system w_1, \ldots, w_n is solvable in a group containing G.

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We have k ≤ n and the presentation complex has itself trivial second homology. This happens when the (n × k)-matrix, whose (i, j)-entry is the signed number of occurances of the letter x_i in ε(w_i), has rank k.

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The case when k = n − 1 and β₁⁽²⁾(Q) = 0, i.e. the first ℓ²-Betti number of the group Q vanishes.

Definition

A group G is called sofic if for all finite subsets $F \subseteq G$ and $\varepsilon > 0$ there is $n \ge 1$ and a mapping $\varphi : F \to \text{Sym}(n)$ such that

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Here

$$d(\sigma,\tau) = \frac{|\{1 \le i \le n \mid \sigma(i) \ne \tau(i)\}|}{n}$$

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is the normalized Hamming length on Sym(n).

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Are all groups sofic?



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Theorem (Elek-Szabo)

Sofic groups are Connes-embeddable.

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More concretely, if $\Gamma = \langle X \mid R \rangle$ is finitely presented, then we look for interesting homomorphisms $\alpha \colon F_X \to \text{Sym}(n)$, such that $\alpha(r)$ is ε -close to 1_n for all $r \in R$

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Proposition

A finitely presented group $\Gamma = \langle X \mid R \rangle$ is sofic if and only if there exists a sequence $(\alpha_n \colon F_X \to \operatorname{Sym}(k_n))_{n \in \mathbb{N}}$

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- For g ∉ ⟨⟨R⟩⟩, the sequence (α_n(g))_{n∈ℕ} of permutations fixes less and less.

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Definition

A group G is called stable, when for all $\varepsilon > 0$, there exists $\delta > 0$, such that every δ -homomorphism from G to Sym(n) is ε -close to a homomorphism.

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Theorem (with Becker and Lubotzky, 2017)

An amenable group is stable if and only if all IRS are limits of finite-index IRS. This happens for example for polycyclic groups, but not for all residually finite groups.

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Corollary (Arzhantseva-Paunescu, 2014)

Almost commuting permutations are close to commuting permutations.

Theorem

A sofic and stable group is residually finite, i.e. its elements can be separated in finite quotients.

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In fact, we have seen an example in the beginning

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So, providing examples of non-residually finite and stable groups, would show also the existence of non-sofic groups

Non-residually finite central extensions

Theorem (Deligne, 1978)

There exist lattices in algebraic groups that admit finite central extensions which are not residually finite anymore.

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If Deligne's lattices satisfies some weak form of stability, then the central extension is not sofic.

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Remark

Currently, there is no residually finite group known *not* to satisfy the weak form of stability relevant in the previous theorem.

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A p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is weakly contained in a family of p.m.p. actions $\Gamma \curvearrowright (X_{\alpha}, \mu_{\alpha})$ if for any partition $X = A_1 \cup \cdots \cup A_n$, any $\varepsilon > 0$ and any $K \subset \Gamma$ finite there exists α and a partition $X_{\alpha} = B_1 \cup \cdots \cup B_n$

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Definition

A group Γ is called *stable in finite actions* if the limit action of each sofic approximation is weakly contained in finite Γ -actions.

Consider a finite central extension

$$1 \rightarrow A \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1$$

classified by $[\alpha] \in \mathrm{H}^2(\Gamma, A)$.

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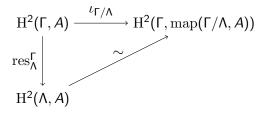
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Consider the Shapiro isomorphism :



In presence of suitable uniform control on the expansion of the second differential in the chain complex of EΓ, we obtain that

 $0 \neq \iota_{(X,\mu)}([\alpha]) \in \mathrm{H}^2(\Gamma, \mathrm{M}(X, A))$

for any action $\Gamma \curvearrowright (X, \mu)$ weakly contained in finite Γ -actions.

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Starting with a sofic approximation of Γ̃, we obtain a non-standard limit action Γ̃ → (X, μ) such that

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with Y = X/A, i.e., $M(Y, A) = M(X, A)^A$.

This implies that the induced sofic approximation of Γ is not weakly contained in finite actions.

$\mathrm{MIP}^* = \mathrm{RE}$

A recent breakthrough result in computer science and quantum information theory by Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen implies that cooperative games with quantum entanglement can compute any recursively enumerable set.

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As a consequence, the Connes Embedding Problem has a negative answer, since a positive answer would imply that the outcome cooperative games with quantum entanglement is computable.

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As a consequence, the Connes Embedding Problem has a negative answer, since a positive answer would imply that the outcome cooperative games with quantum entanglement is computable.

This does not directly lead to a non-Connes-embeddable (hence non-sofic) group, but it is a very promising direction to follow at the moment.

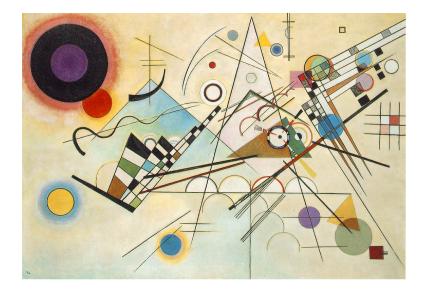
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The closest we can get at the moment is:

Theorem (with Hiroshi Ando, 2023)

There exists a topological group of finite type which is not Connes-embeddable.

Thank you for your attention!



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