High-dimensional expansion and soficity of groups

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From Analysis to Homotopy Theory
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Overview

1. Equations over groups
2. Finitary approximation properties of groups
3. High-dimensional expansion and soficity of groups

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The study of equations like this goes back to:

Bernhard H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc. 18 (1943), 4-11.

The general setting

Each equation

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- When is w surjective?
- Is every self-map of Γ of this form?

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But the automorphism of $\mathbb{Z}/p\mathbb{Z}$ which sends 1 to 2 has order dividing $p - 1$ and hence the order is co-prime to p.

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The resulting effect on fundamental groups is exactly

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Such groups are called Connes-embeddable, more precisely,

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Let $w_1, w_2, \ldots, w_k \in \mathbb{F}_n * G$, and denote by $\varepsilon(w_1), \ldots, \varepsilon(w_k) \in \mathbb{F}_n$ their images under the natural homomorphism $\varepsilon\colon \mathbb{F}_n * G \to \mathbb{F}_n$.

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Consider $X \subset Y \to Y/X$, with Y/X two-dimensional. When is $\pi_1(X) \to \pi_1(Y)$ injective?

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- When the presentation complex is aspherical. Note that in this case the number of equations k can be larger than the number of variables.
- The case when $k = n - 1$ and $\beta_1^{(2)}(Q) = 0$, i.e. the first ℓ^2-Betti number of the group Q vanishes.

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Here

$$d(\sigma, \tau) = \frac{|\{1 \leq i \leq n \mid \sigma(i) \neq \tau(i)\}|}{n}$$

is the normalized Hamming length on $\mathrm{Sym}(n)$.

Approximation by symmetric groups

Question (Gromov)

Are all groups sofic?

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Theorem (Elek-Szabo)

Sofic groups are Connes-embeddable.

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More concretely, if $\Gamma = \langle X \mid R \rangle$ is finitely presented, then we look for interesting homomorphisms $\alpha \colon F_X \to \mathrm{Sym}(n)$, such that $\alpha(r)$ is ε-close to 1_n for all $r \in R$

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- *α_n is a $(1/n)$-homomorphism, and*
- *For $g \notin \langle\langle R \rangle\rangle$, the sequence $(\alpha_n(g))_{n\in\mathbb{N}}$ of permutations fixes less and less.*

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Theorem (with Becker and Lubotzky, 2017)

An amenable group is stable if and only if all IRS are limits of finite-index IRS. This happens for example for polycyclic groups, but not for all residually finite groups.

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Corollary (Arzhantseva-Paunescu, 2014)

Almost commuting permutations are close to commuting permutations.

Why stability?

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$$\Gamma = \langle a, t \mid a^2 = (tat^{-1})a(tat^{-1})^{-1} \rangle.$$

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So, providing examples of non-residually finite and stable groups, would show also the existence of non-sofic groups

Non-residually finite central extensions

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Remark

Currently, there is no residually finite group known *not* to satisfy the weak form of stability relevant in the previous theorem.

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Definition

A group Γ is called *stable in finite actions* if the limit action of each sofic approximation is weakly contained in finite Γ-actions.

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- Consider a finite central extension

$$1 \to A \to \tilde{\Gamma} \to \Gamma \to 1$$

classified by $[\alpha] \in \mathrm{H}^2(\Gamma, A)$.

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- Consider the Shapiro isomorphism :

$$\begin{array}{ccc}
\mathrm{H}^2(\Gamma, A) & \xrightarrow{\iota_{\Gamma/\Lambda}} & \mathrm{H}^2(\Gamma, \mathrm{map}(\Gamma/\Lambda, A)) \\
\Big\downarrow \mathrm{res}^{\Gamma}_{\Lambda} & \nearrow \sim & \\
\mathrm{H}^2(\Lambda, A) & &
\end{array}$$

Idea of the proof – Part III

- In presence of suitable uniform control on the expansion of the second differential in the chain complex of $E\Gamma$, we obtain that

$$0 \neq \iota_{(X,\mu)}([\alpha]) \in \mathrm{H}^2(\Gamma, \mathrm{M}(X, A))$$

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- This implies that the induced sofic approximation of Γ is not weakly contained in finite actions.

$\mathrm{MIP}^* = \mathrm{RE}$

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As a consequence, the Connes Embedding Problem has a negative answer, since a positive answer would imply that the outcome cooperative games with quantum entanglement is computable.

This does not directly lead to a non-Connes-embeddable (hence non-sofic) group, but it is a very promising direction to follow at the moment.

The closest we can get at the moment is:

Theorem (with Hiroshi Ando, 2023)

There exists a topological group of finite type which is not Connes-embeddable.

Thank you for your attention!