On the Interaction between Singular and Sheaf Cohomology

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Motivating Question

For which spaces X is $H^*_{\mathrm{sh}}(X; R) \simeq H^*_{\mathrm{sing}}(X; R)$?

Conventions

- X topological space
- $R \in \mathbb{E}_\infty$-*Ring*
- $\mathcal{D} := Mod_R(Sp)$

The Statement

Def: X is R-cohomologically locally connected iff

$$\forall x \in X \ : \ \operatorname*{colim}_{U \ni x} \tilde{H}^*_{\mathrm{sing}}(U; R) \simeq 0$$

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Thm: X is R-cohomologically locally connected iff

$$\mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{D} \quad ; \quad U \mapsto |C^*_{\mathrm{sing}}(U; R)|$$

is a hypercompletion of $\underline{R} \in \mathrm{Sh}(X; \mathcal{D})$.

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Cor: If R is discrete or X is of finite homotopy dimension TFAE:

- X is R-c.l.c.
- $\forall U \in \mathrm{Open}(X) \;:\; H^*_{\mathrm{sh}}(U; R) \simeq H^*_{\mathrm{sing}}(U; R) \in Ab^*$.
- $\forall U \in \mathrm{Open}(X) \;:\; \Gamma(\underline{R}; U) \simeq |C^*_{\mathrm{sing}}(U; R)| \in \mathbb{E}_\infty\text{-}Ring$.

Central Ingredient

Thm: [Dugger, Hollander, Isaksen 2006] The map

$$\mathrm{Open}(X) \to \mathcal{S} \quad ; \quad U \mapsto \tau(U) := |\mathrm{Sing}_*(U)|$$

satisfies (co)descent w.r.t. to all hypercovers.

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Cor: The map above lifts to a cocontinuous functor

$$\varphi : \mathrm{Sh}(X) \longrightarrow \mathcal{S}_{/\tau(X)}$$

The Proof

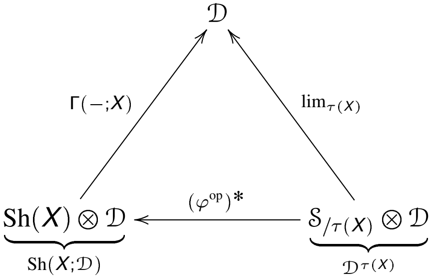
$$\begin{array}{ccc} & \mathcal{D} & \\ \swarrow & & \searrow \\ \mathrm{Sh}(X) \otimes \mathcal{D} & \xrightarrow{\varphi \otimes \mathrm{id}_{\mathcal{D}}} & \mathcal{S}_{/\tau(X)} \otimes \mathcal{D} \end{array}$$

The Proof

$$\begin{array}{ccc} & \mathcal{D} & \\ {\scriptstyle \mathrm{const}_{\mathrm{Sh}(X)}} \swarrow & & \searrow {\scriptstyle \mathrm{const}_{\tau(X)}} \\ \underbrace{\mathrm{Sh}(X) \otimes \mathcal{D}}_{\mathrm{Sh}(X;\mathcal{D})} & \xrightarrow{\varphi \otimes \mathrm{id}_{\mathcal{D}}} & \underbrace{\mathcal{S}_{/\tau(X)} \otimes \mathcal{D}}_{\mathcal{D}^{\tau(X)}} \end{array}$$

$$\underline{R}_{\mathrm{Sh}(X)} \longmapsto \underline{R}_{\tau(X)}$$

The Proof



$\rightsquigarrow$ unit morphism $\eta_{\underline{R}} : \underline{R}_{\mathrm{Sh}(X)} \longrightarrow (\underline{R}_{\tau(X)} \circ \varphi^{op})$

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- DK $\Rightarrow$ $(\underline{R}_{\tau(X)} \circ \varphi^{\mathrm{op}}) \simeq \lim_{\tau(-)} R \simeq |C^*_{\mathrm{sing}}(-; R)|$.

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- Thm. $\Rightarrow$ $(\underline{R}_{\tau(X)} \circ \varphi^{\mathrm{op}})$ is a hypersheaf.
- Check iso. on stalks of homotopy sheaves:

$$\pi_* \mathrm{cofib}(\eta_{\underline{R},x}) \simeq \operatorname*{colim}_{U \ni x} \tilde{H}^*_{\mathrm{sing}}(U; R)$$

Mixed Künneth

Prop: If X is locally contractible and Y is any space then

$$\Gamma(\underline{R}_{X \times Y}; X \times Y) \simeq \Gamma(\underline{R}_Y; Y)^{\tau(X)}$$

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Cor: The projection $I \times Y \to Y$ induces an equivalence

$$\Gamma(\underline{R}_{I\times Y}; I \times Y) \simeq \Gamma(\underline{R}_Y; Y)$$