Disentangling orbifold data

From Analysis to Homotopy Theory
A conference in honor of Ulrich Bunke's 60th birthday
Alfried-Krupp Wissenschaftskolleg, Greifswald, Germany
May 13-17, 2024

Katrin Wendland
Trinity College Dublin

Plan:
1. Motivation: orbifolding
2. An application of the equivariant index theorem?
3. Appreciating q-series
4. Regularized elliptic genera for simple surface singularities

[Hou/W] *The complex elliptic genus for simple surface singularities*, work in progress, based on Yuhang Hou's 2021 PhD thesis *Elliptic genera of ADE type singularities*, Freiburg

[W17] *Hodge-elliptic genera and how they govern K3 theories*, Commun. Math. Phys. 368 (2019), 187-221; arXiv:1705.09904 [hep-th]

Orbifolding partition functions

CFT elliptic genus of a toroidal CFT in two complex dimensions:

$\mathbb{H}$: subspace of space of states (a vector space), $J_0, \widetilde{J}_0, L_0^{\mathrm{top}} \in \mathrm{End}(\mathbb{H})$,

$\tau,\ z \in \mathbb{C},\ \mathrm{Im}(\tau) > 0,\ q = \exp(2\pi i\tau),\ y = \exp(2\pi iz);$

.

$$
{}_1\square_1 = \mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}}\right)
= \underbrace{\prod_{n=1}^{\infty}(1-q^n)^{-4}}_{\eta(\tau)^{-4}} \underbrace{\prod_{n=1}^{\infty}(1-q^{n-1}y)^2(1-q^n y^{-1})^2 \cdot y^{-1}}_{(\vartheta_1(\tau,z)/\eta(\tau))^2} \cdot (1-2+1)
$$

Orbifolding partition functions

CFT elliptic genus of a toroidal CFT in two complex dimensions:

$\mathbb{H}$: subspace of space of states (a vector space), $J_0, \widetilde{J}_0, L_0^{\mathrm{top}} \in \mathrm{End}(\mathbb{H})$,

$$\tau,\ z \in \mathbb{C},\ \mathrm{Im}(\tau) > 0,\ q = \exp(2\pi i\tau),\ y = \exp(2\pi i z);$$

$$\{1, \iota\} \cong \mathbb{Z}_2, \qquad \mathbb{H}^{\mathbb{Z}_2} = \tfrac{1}{2}(1+\iota)\mathbb{H}, \qquad \mathcal{E}^{\mathbb{Z}_2} = \tfrac{1}{2}\Big({}_1\underset{1}{\square} + {}_\iota\underset{1}{\square}\Big).$$

$$\begin{aligned} {}_1\underset{1}{\square} &= \mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}}\right) \\ &= \underbrace{\prod_{n=1}^{\infty}(1-q^n)^{-4}}_{\eta(\tau)^{-4}} \underbrace{\prod_{n=1}^{\infty}(1-q^{n-1}y)^2(1-q^n y^{-1})^2 \cdot y^{-1}}_{(\vartheta_1(\tau,z)/\eta(\tau))^2} \cdot (1-2+1) \end{aligned}$$

Orbifolding partition functions

CFT elliptic genus of a toroidal CFT in two complex dimensions:

$\mathbb{H}$: subspace of space of states (a vector space), $J_0, \widetilde{J}_0, L_0^{\text{top}} \in \text{End}(\mathbb{H})$,

$\tau,\, z \in \mathbb{C},\ \text{Im}(\tau) > 0,\ q = \exp(2\pi i\tau),\ y = \exp(2\pi iz);$

$$\{1, \iota\} \cong \mathbb{Z}_2, \qquad \mathbb{H}^{\mathbb{Z}_2} = \tfrac{1}{2}(1+\iota)\mathbb{H}, \qquad \mathcal{E}^{\mathbb{Z}_2} = \tfrac{1}{2}\Big(1\,\underset{1}{\square} + \iota\,\underset{1}{\square}\Big).$$

$$\iota\,\underset{1}{\square} = \text{tr}_{\mathbb{H}}\left((-1)^{J_0 - \widetilde{J}_0} y^{J_0 - 1} q^{L_0^{\text{top}}} \iota\right)$$

$$= \underbrace{\prod_{n=1}^{\infty}(1+q^n)^{-4}}_{(\vartheta_2(\tau,0)/2\eta(\tau))^{-2}}\ \underbrace{\prod_{n=1}^{\infty}(1+q^{n-1}y)^2(1+q^n y^{-1})^2 \cdot y^{-1}}_{(\vartheta_2(\tau,z)/\eta(\tau))^2} \cdot (1+2+1)$$

Orbifolding partition functions

CFT elliptic genus of a toroidal CFT in two complex dimensions:

$\mathbb{H}$: subspace of space of states (a vector space), $J_0, \widetilde{J}_0, L_0^{\mathrm{top}} \in \mathrm{End}(\mathbb{H})$,

$$\tau,\, z \in \mathbb{C},\ \mathrm{Im}(\tau) > 0,\ q = \exp(2\pi i\tau),\ y = \exp(2\pi i z);$$

$$\{1, \iota\} \cong \mathbb{Z}_2, \qquad \mathbb{H}^{\mathbb{Z}_2} = \tfrac{1}{2}(1+\iota)\mathbb{H}, \qquad \mathcal{E}^{\mathbb{Z}_2} = \tfrac{1}{2}\Big({}_{1}\underset{1}{\square} + {}_{\iota}\underset{1}{\square}\Big).$$

$$
\begin{aligned}
{}_{\iota}\underset{1}{\square} &= \mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}}\,\iota\right)\\
&= \underbrace{\prod_{n=1}^{\infty}(1+q^n)^{-4}}_{(\vartheta_2(\tau,0)/2\eta(\tau))^{-2}}\ \underbrace{\prod_{n=1}^{\infty}(1+q^{n-1}y)^2(1+q^n y^{-1})^2\cdot y^{-1}}_{(\vartheta_2(\tau,z)/\eta(\tau))^2}\cdot(1+2+1)
\end{aligned}
$$

$${}_{1}\underset{\iota}{\square}(\tau,z) := \exp(-2\pi i z^2/\tau)\cdot {}_{\iota}\underset{1}{\square}(-1/\tau, z/\tau) = 16\left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2,$$

$${}_{\iota}\underset{\iota}{\square}(\tau,z) := {}_{1}\underset{\iota}{\square}(\tau+1,z) \quad = \quad 16\left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2,$$

$$\tfrac{1}{2}\Big({}_{1}\underset{1}{\square} + {}_{\iota}\underset{1}{\square} + {}_{1}\underset{\iota}{\square} + {}_{\iota}\underset{\iota}{\square}\Big) = \mathcal{E}^{\mathrm{K3}}.$$

Geometric orbifolds

M: a Calabi-Yau D-fold, G: a symmetry group of M, that is,
$G \subset \mathrm{SU}(D)$: a finite group, acting holomorphically, isometrically on M,
such that the holomorphic volume form is preserved.

Then $X := \widetilde{M/G}$ is a Calabi-Yau D-fold, a G-orbifold of M.

Expect: existence of orbifold conformal field theories
and geometric interpretations on orbifold Calabi-Yaus.

Geometric orbifolds: orbifold constructions of K3

M: a Calabi-Yau D-fold, G: a symmetry group of M, that is,
$G \subset \mathrm{SU}(D)$: a finite group, acting holomorphically, isometrically on M,
such that the holomorphic volume form is preserved.

Then $X := \widetilde{M/G}$ is a Calabi-Yau D-fold, a G-orbifold of M.

Expect: existence of orbifold conformal field theories
and geometric interpretations on orbifold Calabi-Yaus.

Kummer-like constructions:

Let $M = T = \mathbb{C}^2/L$, $L \subset \mathbb{C}^2$ a lattice of rank 4; G a finite group with
$G \subset \mathrm{SU}(2)$, $1 < |G|$, $GL = L$; then $\widetilde{T/G}$ is a K3 surface.

Geometric orbifolds: orbifold constructions of K3

M: a Calabi-Yau D-fold, G: a symmetry group of M, that is,
$G \subset \mathrm{SU}(D)$: a finite group, acting holomorphically, isometrically on M, such that the holomorphic volume form is preserved.

Then $X := \widetilde{M/G}$ is a Calabi-Yau D-fold, a G-orbifold of M.

Expect: existence of orbifold conformal field theories and geometric interpretations on orbifold Calabi-Yaus.

Kummer-like constructions:

Let $M = T = \mathbb{C}^2/L$, $L \subset \mathbb{C}^2$ a lattice of rank 4; G a finite group with $G \subset \mathrm{SU}(2)$, $1 < |G|$, $GL = L$; then $\widetilde{T/G}$ is a K3 surface.

[Fujiki88]: G is cyclic of order $2, 3, 4$ or 6, or binary dihedral of order 8 or 12, or binary tetrahedral.

Building the bridge: K3 theories

Result (generalizing **[Nahm/W99, W00]**)
For a complex 2-torus or a K3 surface M and $G \subset \mathrm{SU}(2)$ as above,
let $X_0 := M/G$ and $X := \widetilde{X_0}$;
then we have definitions of TOROIDAL THEORIES and K3 THEORIES, such that the SCFT associated with X
is the G-orbifold of the theory associated with M.

Building the bridge: K3 theories

Result (generalizing **[Nahm/W99, W00]**)
For a complex 2-torus or a K3 surface M and $G \subset \mathrm{SU}(2)$ as above,
let $X_0 := M/G$ and $X := \widetilde{X_0}$;
then we have definitions of TOROIDAL THEORIES and K3 THEORIES,
such that the SCFT associated with X
is the G-orbifold of the theory associated with M.

The CFT ELLIPTIC GENUS of a SCFT associated with X
with topologically half-twisted model $\mathbb{H}$ is

$$\mathcal{E}(\tau, z) := \mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}}\right)$$

Building the bridge: K3 theories

Result (generalizing **[Nahm/W99, W00]**)
For a complex 2-torus or a K3 surface M and $G \subset \mathrm{SU}(2)$ as above,
let $X_0 := M/G$ and $X := \widetilde{X_0}$;
then we have definitions of TOROIDAL THEORIES and K3 THEORIES,
such that the SCFT associated with X
is the G-orbifold of the theory associated with M.

The CFT ELLIPTIC GENUS of a SCFT associated with X
with topologically half-twisted model $\mathbb{H}$ is

$$\mathcal{E}(\tau, z) := \mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0 - \widetilde{J}_0} y^{J_0 - 1} q^{L_0^{\mathrm{top}}}\right),$$

a WEAK JACOBI FORM of weight 0 and index 1 on $SL_2(\mathbb{Z})$:

$$\mathcal{E}(\tau+1, z) = \mathcal{E}(\tau, z), \qquad \mathcal{E}(-1/\tau, z/\tau) = \exp(2\pi i z^2/\tau)\mathcal{E}(\tau, z),$$
$$\mathcal{E}(\tau, z+1) = \mathcal{E}(\tau, z), \qquad \mathcal{E}(\tau, z+\tau) = q^{-1} y^{-2}\mathcal{E}(\tau, z).$$

Building the bridge: K3 theories

Result (generalizing [Nahm/W99, W00])
For a complex 2-torus or a K3 surface M and $G \subset \mathrm{SU}(2)$ as above,
let $X_0 := M/G$ and $X := \widetilde{X_0}$;
then we have definitions of TOROIDAL THEORIES and K3 THEORIES,
such that the SCFT associated with X
is the G-orbifold of the theory associated with M.

The CFT ELLIPTIC GENUS of a SCFT associated with X
with topologically half-twisted model $\mathbb{H}$ is

$$\mathcal{E}(\tau, z) := \mathrm{tr}_{\mathbb{H}} \left((-1)^{J_0 - \widetilde{J}_0} y^{J_0 - 1} q^{L_0^{\mathrm{top}}} \right),$$

a WEAK JACOBI FORM of weight 0 and index 1 on $SL_2(\mathbb{Z})$:

$$\mathcal{E}(\tau+1, z) = \mathcal{E}(\tau, z), \qquad \mathcal{E}(-1/\tau, z/\tau) = \exp(2\pi i z^2/\tau)\mathcal{E}(\tau, z),$$
$$\mathcal{E}(\tau, z+1) = \mathcal{E}(\tau, z), \qquad \mathcal{E}(\tau, z+\tau) = q^{-1} y^{-2} \mathcal{E}(\tau, z).$$

$\mathcal{E}$ agrees with the COMPLEX ELLIPTIC GENUS of X.

The guiding question

For K3 surfaces and/or K3 theories
that are obtained as toroidal orbifolds,
can we disentangle the orbifold data
to determine the contributions to the elliptic genus
from each resolved simple surface singularity?

The guiding question

For K3 surfaces and/or K3 theories
that are obtained as toroidal orbifolds,
can we disentangle the orbifold data
to determine the contributions to the elliptic genus
from each resolved simple surface singularity?

From now on:

$\Gamma \subset \mathrm{SU}(2)$ is a finite subgroup,
i.e. cyclic (A_k) or binary dihedral (D_k) or binary Platonic (E_k);
Γ also denotes $\widetilde{\mathbb{C}^2/\Gamma}$.

The complex elliptic genus in equivariant index format

Definition [Hirzebruch88, Witten88]

COMPLEX ELLIPTIC GENUS $\mathcal{E}(M;\tau,z)$ of

M: a compact complex D-dimensional manifold, with $T := T^{1,0}M$ its holomorphic tangent bundle;

using the splitting principle, $c(T) = \prod_{j=1}^{D}(1+x_j)$,

$$\mathcal{E}(M;\tau,z) \;:=\; y^{-D/2}\int_M \prod_{j=1}^{D}\Bigg[\underbrace{\frac{x_j}{1-e^{-x_j}}}_{\mathrm{Td}}\underbrace{\left(1-ye^{-x_j}\right)}_{\mathrm{ch}(\Lambda_{-y}T^*)}\cdot \prod_{n=1}^{\infty}\frac{(1-ye^{-x_j}q^n)(1-y^{-1}e^{x_j}q^n)}{(1-e^{-x_j}q^n)(1-e^{x_j}q^n)}\Bigg]$$

where for any bundle $E\to M$, $\Lambda_x E := \bigoplus_{k=0}^{\infty} x^k\Lambda^k E$, $S_x E := \bigoplus_{k=0}^{\infty} x^k S^k E$

The complex elliptic genus in equivariant index format

Definition **[Hirzebruch88, Witten88]**

COMPLEX ELLIPTIC GENUS $\mathcal{E}(M;\tau,z)$ of

M: a compact complex D-dimensional manifold, with

$T := T^{1,0}M$ its holomorphic tangent bundle;

using the splitting principle, $c(T) = \prod_{j=1}^{D}(1+x_j)$,

$$\mathcal{E}(M;\tau,z) \quad := \quad y^{-D/2}\int_M \prod_{j=1}^{D}\Bigg[\underbrace{\frac{x_j}{1-e^{-x_j}}}_{\mathrm{Td}}\left(1-ye^{-x_j}\right)\cdot \prod_{n=1}^{\infty}\frac{\overbrace{\left(1-ye^{-x_j}q^n\right)}^{\mathrm{ch}\left(\Lambda_{-yq^n}T^*\right)}\overbrace{\left(1-y^{-1}e^{x_j}q^n\right)}^{\mathrm{ch}\left(\Lambda_{-y^{-1}q^n}T\right)}}{\underbrace{\left(1-e^{-x_j}q^n\right)}_{\mathrm{ch}\left(S_{q^n}T^*\right)}\underbrace{\left(1-e^{x_j}q^n\right)}_{\mathrm{ch}\left(S_{q^n}T\right)}}\Bigg]$$

where for any bundle $E \to M$, $\Lambda_x E := \bigoplus_{k=0}^{\infty} x^k \Lambda^k E$, $S_x E := \bigoplus_{k=0}^{\infty} x^k S^k E$

The complex elliptic genus in equivariant index format

Definition **[Hirzebruch88, Witten88]**
Complex elliptic genus $\mathcal{E}(M;\tau,z)$ of
M: a compact complex D-dimensional manifold, with
$T := T^{1,0}M$ its holomorphic tangent bundle;
using the splitting principle, $c(T) = \prod_{j=1}^{D}(1+x_j)$,

$$\mathcal{E}(M;\tau,z) := y^{-D/2}\int_M \prod_{j=1}^{D}\left[\frac{x_j}{1-e^{-x_j}}\left(1-ye^{-x_j}\right)\cdot \prod_{n=1}^{\infty}\frac{(1-ye^{-x_j}q^n)(1-y^{-1}e^{x_j}q^n)}{(1-e^{-x_j}q^n)(1-e^{x_j}q^n)}\right]$$

$$= \int_M \prod_{j=1}^{D}\left[x_j\frac{\vartheta_1(\tau,z-x_j)}{\vartheta_1(\tau,-x_j)}\right] \quad \in \quad y^{-D/2}\cdot\mathbb{Z}[[q,y^{\pm 1}]],$$

where for any bundle $E \to M$, $\Lambda_x E := \bigoplus_{k=0}^{\infty} x^k\Lambda^k E$, $S_x E := \bigoplus_{k=0}^{\infty} x^k S^k E$

The equivariant index theorem

Equivariant index theorem [Atiyah/Bott67, Atiyah/Singer68]
(see also [Hirzebruch/Berger/Jung92, Waelder08])

M: a compact complex D-dimensional manifold,
C: a compact topological group, acting holomorphically on M, $g \in C$.

With $M^g = \dot\cup_K M_g^K$ the decomposition into connected components,
$T_{|M_g^K} = \oplus_\lambda N_\lambda^K$ the eigenbundle decomposition
with respect to the action of g, with eigenvalues $\lambda = e^{2\pi i \zeta_\lambda}$,
and according to the splitting principle, for the total Chern class $c(N_\lambda^K)$,

$$c(N_\lambda^K) = \prod_{j=1}^{r_\lambda^K} (1 + x_\lambda^j):$$

the corresponding EQUIVARIANT ELLIPTIC GENUS is

$$\mathcal{E}^g(M; z, \tau) = \sum_K \int_{M_g^K} \prod_{m=1}^{r_1^K} x_1^m \prod_\lambda \prod_{j=1}^{r_\lambda^K} \frac{\vartheta_1(\tau, z + \zeta_\lambda - x_\lambda^j)}{\vartheta_1(\tau, \zeta_\lambda - x_\lambda^j)}.$$

Application to resolved simple surface singularities

Example [Hou/W]

With the $\mathbb{C}^*$-action induced by $\mathbb{C}^* \hookrightarrow \mathrm{GL}_2(\mathbb{C})$, $\xi \mapsto \xi \cdot \mathrm{id}$, for $M = \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$:

if $\xi \in \mathbb{C}^*$, $\xi = e^{2\pi i \zeta}, \xi \neq \pm 1$: $\quad M^\xi \cong \mathbb{CP}^1$, $\quad T_{|\mathbb{CP}^1} = N_1 \oplus N_{\xi^2}$,

$$\begin{aligned}
\mathcal{E}^\xi(A_1;\tau,z) &= \int_{\mathbb{CP}^1} x_1 \frac{\vartheta_1(\tau,z-x_1)\cdot\vartheta_1(\tau,z+2\zeta-x_2)}{\vartheta_1(\tau,-x_1)\cdot\vartheta_1(\tau,2\zeta-x_2)} \\
&= \frac{\partial_z\vartheta_1(\tau,z)\cdot\vartheta_1(\tau,z+2\zeta)-\vartheta_1(\tau,z)\cdot\partial_z\vartheta_1(\tau,z+2\zeta)}{\pi\vartheta_1(\tau,2\zeta)\cdot\eta(\tau)^3} \\
&\quad + \frac{\vartheta_1(\tau,z)\cdot\vartheta_1(\tau,z+2\zeta)\cdot\partial_z\vartheta_1(\tau,2\zeta)}{\pi\vartheta_1(\tau,2\zeta)^2\cdot\eta(\tau)^3}
\end{aligned}$$

Application to resolved simple surface singularities

Example [Hou/W]

With the $\mathbb{C}^*$-action induced by $\mathbb{C}^* \hookrightarrow \mathrm{GL}_2(\mathbb{C})$, $\xi \mapsto \xi \cdot \mathrm{id}$, for $M = \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$:

if $\xi \in \mathbb{C}^*$, $\xi = e^{2\pi i \zeta}, \xi \neq \pm 1$: $\quad M^\xi \cong \mathbb{CP}^1$, $\quad T_{|\mathbb{CP}^1} = N_1 \oplus N_{\xi^2}$,

$$\begin{aligned}
\mathcal{E}^\xi(A_1; \tau, z) &= \int_{\mathbb{CP}^1} x_1 \frac{\vartheta_1(\tau, z - x_1) \cdot \vartheta_1(\tau, z + 2\zeta - x_2)}{\vartheta_1(\tau, -x_1) \cdot \vartheta_1(\tau, 2\zeta - x_2)} \\
&= \frac{\partial_z \vartheta_1(\tau, z) \cdot \vartheta_1(\tau, z + 2\zeta) - \vartheta_1(\tau, z) \cdot \partial_z \vartheta_1(\tau, z + 2\zeta)}{\pi \vartheta_1(\tau, 2\zeta) \cdot \eta(\tau)^3} \\
&\quad + \frac{\vartheta_1(\tau, z) \cdot \vartheta_1(\tau, z + 2\zeta) \cdot \partial_z \vartheta_1(\tau, 2\zeta)}{\pi \vartheta_1(\tau, 2\zeta)^2 \cdot \eta(\tau)^3}
\end{aligned}$$

Result [Hou/W]

Similar formulas for each of the ADE-type singularities $M = \widetilde{\mathbb{C}^2/\Gamma}$, $\Gamma \subset \mathrm{SU}(2)$, yielding a $\mathbb{C}^*$-equivariant elliptic genus $\mathcal{E}^\xi(\Gamma; \tau, z)$ for all $\xi = e^{2\pi i \zeta}$ with $\xi^N \neq 1$ for all divisors N of $|\Gamma|$.

Topologically half-twisted models for ADE singularities

Fixing $\Gamma \subset \mathrm{SU}(2)$ of type A_k, D_k, E_k for $\widetilde{\mathbb{C}^2/\Gamma}$:

Can we associate a topologically half-twisted model

and a conformal field theoretic elliptic genus?

Topologically half-twisted models for ADE singularities

Fixing $\Gamma \subset \mathrm{SU}(2)$ of type A_k, D_k, E_k for $\widetilde{\mathbb{C}^2/\Gamma}$:

Can we associate a topologically half-twisted model

and a conformal field theoretic elliptic genus?

For $M = \mathbb{C}^2$:

$$\mathbb{H}^{\mathbb{C}^2} \cong \mathbb{C}\left[\left\{a_n^\ell, b_m^\ell\right\}_{\substack{\ell\in\{1,2\},\\ m\in\mathbb{N}, n\in\mathbb{N}_{>0}}}\right] \otimes \bigwedge_{\ell\in\{1,2\}, m\in\mathbb{N}, n\in\mathbb{N}_{>0}} \left(\mathbb{C}\left\{\varphi_m^\ell\right\} \wedge \mathbb{C}\left\{\psi_n^\ell\right\}\right),$$

$k\in\{1,2\}$, $m\in\mathbb{N}, n\in\mathbb{N}_{>0}$	a_n^ℓ	b_m^ℓ	φ_m^ℓ	ψ_n^ℓ
$[L_0^{\mathrm{top}}, \cdot]$	$n\cdot$	$m\cdot$	$m\cdot$	$n\cdot$
$[J_0, \cdot]$	0	0	$1\cdot$	$(-1)\cdot$
$[\widetilde{J}_0, \cdot]$	0	0	0	0

with L_0^{top}, J_0, $\widetilde{J}_0$ all vanishing on $1 \in \mathbb{H}$.

Then $\mathrm{tr}_{\mathbb{H}^{\mathbb{C}^2}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}}\right)$ is ill-defined.

Topologically half-twisted models for ADE singularities

Fixing $\Gamma \subset \mathrm{SU}(2)$ of type A_k, D_k, E_k for $\widetilde{\mathbb{C}^2/\Gamma}$:

Can we associate a topologically half-twisted model

and a conformal field theoretic elliptic genus?

For $M = \mathbb{C}^2$:

$$\mathbb{H}^{\mathbb{C}^2} \cong \mathbb{C}\left[\left\{a_n^\ell, b_m^\ell\right\}_{\substack{\ell\in\{1,2\},\\ m\in\mathbb{N}, n\in\mathbb{N}_{>0}}}\right] \otimes \bigwedge_{\ell\in\{1,2\}, m\in\mathbb{N}, n\in\mathbb{N}_{>0}} \left(\mathbb{C}\left\{\varphi_m^\ell\right\} \wedge \mathbb{C}\left\{\psi_n^\ell\right\}\right),$$

$k\in\{1,2\}$, $m\in\mathbb{N}, n\in\mathbb{N}_{>0}$	a_n^ℓ	b_m^ℓ	φ_m^ℓ	ψ_n^ℓ
$[L_0^{\mathrm{top}}, \cdot]$	$n\cdot$	$m\cdot$	$m\cdot$	$n\cdot$
$[J_0, \cdot]$	0	0	$1\cdot$	$(-1)\cdot$
$[\widetilde{J}_0, \cdot]$	0	0	0	0
$\xi_\bullet(\cdot)$	$\xi^{-1}\cdot$	$\xi\cdot$	$\xi\cdot$	$\xi^{-1}\cdot$

with L_0^{top}, J_0, $\widetilde{J}_0$ all vanishing on $1\in \mathbb{H}$

and $\mathbb{C}^* \hookrightarrow \mathrm{GL}_2(\mathbb{C})$, $\xi \mapsto \xi\cdot\mathrm{id}$.

Then $\mathrm{tr}_{\mathbb{H}^{\mathbb{C}^2}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}}\right)$ is ill-defined,

but $\mathrm{tr}_{\mathbb{H}^{\mathbb{C}^2}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}} \xi_\bullet\right)$ is well-defined.

The $\mathbb{C}^*$-equivariant elliptic genera for ADE singularities

$\forall \tau \in \mathbb{C}$ with $\operatorname{Im}(\tau) > 0$, $z, \zeta \in \mathbb{C}$: $\qquad q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$, $\xi := e^{2\pi i\zeta}$:

$$\begin{aligned}\mathcal{E}(\mathbb{H}^{\mathbb{C}^2}; \tau, z, \zeta) &:= \operatorname{tr}_{\mathbb{H}^{\mathbb{C}^2}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-1} q^{L_0^{\mathrm{top}}} \xi_\bullet\right)\\ &= y^{-1}\prod_{n=1}^{\infty}\frac{\left(1-q^{n-1}\xi y\right)^2\left(1-q^n(\xi y)^{-1}\right)^2}{\left(1-q^{n-1}\xi\right)^2\left(1-q^n\xi^{-1}\right)^2} = \frac{\vartheta_1(\tau,\zeta+z)^2}{\vartheta_1(\tau,\zeta)^2},\end{aligned}$$

The $\mathbb{C}^*$-equivariant elliptic genera for ADE singularities

$\forall \tau \in \mathbb{C}$ with $\operatorname{Im}(\tau) > 0$, $z, \zeta \in \mathbb{C}$: $\quad q := e^{2\pi i \tau}$, $y := e^{2\pi i z}$, $\xi := e^{2\pi i \zeta}$:

$$\begin{aligned} \mathcal{E}(\mathbb{H}^{\mathbb{C}^2}; \tau, z, \zeta) &:= \operatorname{tr}_{\mathbb{H}^{\mathbb{C}^2}} \left((-1)^{J_0 - \widetilde{J}_0} y^{J_0 - 1} q^{L_0^{\text{top}}} \xi_\bullet \right) \\ &= y^{-1} \prod_{n=1}^{\infty} \frac{\left(1 - q^{n-1}\xi y\right)^2 \left(1 - q^n (\xi y)^{-1}\right)^2}{(1 - q^{n-1}\xi)^2 (1 - q^n \xi^{-1})^2} = \frac{\vartheta_1(\tau, \zeta + z)^2}{\vartheta_1(\tau, \zeta)^2}, \end{aligned}$$

$$\mathcal{E}(A_k; \tau, z, \zeta) = \tfrac{1}{k+1} \sum_{m,n=0}^{k} \frac{\vartheta_1(\tau, z + \zeta + \frac{m+n\tau}{k+1}) \cdot \vartheta_1(\tau, z + \zeta - \frac{m+n\tau}{k+1})}{\vartheta_1(\tau, \zeta + \frac{m+n\tau}{k+1}) \cdot \vartheta_1(\tau, \zeta - \frac{m+n\tau}{k+1})},$$

previously found by **[Harvey/Lee/Murthy15]**.

The $\mathbb{C}^*$-equivariant elliptic genera for ADE singularities

$\forall \tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$, $z, \zeta \in \mathbb{C}$: $q := e^{2\pi i \tau}$, $y := e^{2\pi i z}$, $\xi := e^{2\pi i \zeta}$:

$$\mathcal{E}(\mathbb{H}^{\mathbb{C}^2}; \tau, z, \zeta) := \mathrm{tr}_{\mathbb{H}^{\mathbb{C}^2}} \left((-1)^{J_0 - \widetilde{J}_0} y^{J_0 - 1} q^{L_0^{\mathrm{top}}} \xi_\bullet \right) = y^{-1} \prod_{n=1}^{\infty} \frac{\left(1 - q^{n-1}\xi y\right)^2 \left(1 - q^n (\xi y)^{-1}\right)^2}{(1 - q^{n-1}\xi)^2 (1 - q^n \xi^{-1})^2} = \frac{\vartheta_1(\tau, \zeta + z)^2}{\vartheta_1(\tau, \zeta)^2},$$

$$\mathcal{E}(A_k; \tau, z, \zeta) = \frac{1}{k+1} \sum_{m,n=0}^{k} \frac{\vartheta_1(\tau, z + \zeta + \frac{m+n\tau}{k+1}) \cdot \vartheta_1(\tau, z + \zeta - \frac{m+n\tau}{k+1})}{\vartheta_1(\tau, \zeta + \frac{m+n\tau}{k+1}) \cdot \vartheta_1(\tau, \zeta - \frac{m+n\tau}{k+1})},$$

previously found by **[Harvey/Lee/Murthy15]**.

By **[Hou/W]**:

$$\mathcal{E}(D_k; \tau, z, \zeta) = \tfrac{1}{2}\mathcal{E}(A_{2k-5}; \tau, z, \zeta) + \mathcal{E}(A_3; \tau, z, \zeta) - \tfrac{1}{2}\mathcal{E}(A_1; \tau, z, \zeta),$$

$$\mathcal{E}(E_6; \tau, z, \zeta) = \mathcal{E}(A_5; \tau, z, \zeta) + \tfrac{1}{2}\mathcal{E}(A_3; \tau, z, \zeta) - \tfrac{1}{2}\mathcal{E}(A_1; \tau, z, \zeta),$$

$$\mathcal{E}(E_7; \tau, z, \zeta) = \tfrac{1}{2}\mathcal{E}(A_7; \tau, z, \zeta) + \tfrac{1}{2}\mathcal{E}(A_5; \tau, z, \zeta) + \tfrac{1}{2}\mathcal{E}(A_3; \tau, z, \zeta) - \tfrac{1}{2}\mathcal{E}(A_1; \tau, z, \zeta),$$

$$\mathcal{E}(E_8; \tau, z, \zeta) = \tfrac{1}{2}\mathcal{E}(A_9; \tau, z, \zeta) + \tfrac{1}{2}\mathcal{E}(A_5; \tau, z, \zeta) + \tfrac{1}{2}\mathcal{E}(A_3; \tau, z, \zeta) - \tfrac{1}{2}\mathcal{E}(A_1; \tau, z, \zeta).$$

The $\mathbb{C}^*$-equivariant elliptic genera for ADE singularities

$$\mathcal{E}(A_k;\tau,z,\zeta)=\frac{1}{k+1}\sum_{m,n=0}^{k}\frac{\vartheta_1(\tau,z+\zeta+\frac{m+n\tau}{k+1})\cdot\vartheta_1(\tau,z+\zeta-\frac{m+n\tau}{k+1})}{\vartheta_1(\tau,\zeta+\frac{m+n\tau}{k+1})\cdot\vartheta_1(\tau,\zeta-\frac{m+n\tau}{k+1})},$$

previously found by **[Harvey/Lee/Murthy15]**.

By **[Hou/W]**:

$$\mathcal{E}(D_k;\tau,z,\zeta)=\tfrac12\mathcal{E}(A_{2k-5};\tau,z,\zeta)+\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac12\mathcal{E}(A_1;\tau,z,\zeta),$$
$$\mathcal{E}(E_6;\tau,z,\zeta)=\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac12\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac12\mathcal{E}(A_1;\tau,z,\zeta),$$
$$\mathcal{E}(E_7;\tau,z,\zeta)=\tfrac12\mathcal{E}(A_7;\tau,z,\zeta)+\tfrac12\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac12\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac12\mathcal{E}(A_1;\tau,z,\zeta),$$
$$\mathcal{E}(E_8;\tau,z,\zeta)=\tfrac12\mathcal{E}(A_9;\tau,z,\zeta)+\tfrac12\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac12\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac12\mathcal{E}(A_1;\tau,z,\zeta).$$

Result **[Hou/W]**

For each of the ADE-type singularities $M=\widetilde{\mathbb{C}^2/\Gamma}$, $\Gamma\subset \mathrm{SU}(2)$,
and all $\xi=e^{2\pi i\zeta}$ with $\xi^N\neq 1$ for all divisors N of $|\Gamma|$:

$$\mathcal{E}^{\xi}(\Gamma;\tau,z)=\mathcal{E}(\Gamma;\tau,z,\zeta).$$

Regularizing the elliptic genera of ADE singularities

Inspired by **[Dixon/Harvey/Vafa/Witten85]**, for a Calabi-Yau D-fold M and $G \subset \mathrm{SU}(D)$ a finite group acting on M, such that the Calabi-Yau structure is preserved, let $S \subset M/G$ denote the set of singular points in M/G, which we assume to be discrete;
then for each $s \in S$, one can define an

orbifold Euler characteristic $\chi^{\mathrm{reg}}(s) \in \mathbb{Q}$, such that

$$\chi(\widetilde{M/G}) = \frac{1}{|G|}\chi(M) + \sum_{s\in S} \chi^{\mathrm{reg}}(s).$$

Regularizing the elliptic genera of ADE singularities

Inspired by **[Dixon/Harvey/Vafa/Witten85]**, for a Calabi-Yau D-fold M and $G \subset \mathrm{SU}(D)$ a finite group acting on M, such that the Calabi-Yau structure is preserved, let $S \subset M/G$ denote the set of singular points in M/G, which we assume to be discrete;
then for each $s \in S$, one can define an

ORBIFOLD EULER CHARACTERISTIC $\chi^{\mathrm{reg}}(s) \in \mathbb{Q}$, such that

$$\chi(\widetilde{M/G}) = \frac{1}{|G|}\chi(M) + \sum_{s \in S} \chi^{\mathrm{reg}}(s).$$

Result **[Hou/W]**
For any finite subgroup $\Gamma \subset \mathrm{SU}(2)$, set

$$\mathcal{E}^{\mathrm{reg}}(\Gamma; \tau, z, \zeta) := \mathcal{E}(\Gamma; \tau, z, \zeta) - \frac{1}{|\Gamma|}\mathcal{E}(\mathbb{H}^{\mathbb{C}^2}; \tau, z, \zeta).$$

With M a compact Calabi-Yau 2-fold and $G \subset \mathrm{SU}(2)$ a finite group that acts on M preserving the Calabi-Yau structure, if S denotes the set of singular points in M/G, then let $\Gamma_s \subset \mathrm{SU}(2)$ denote the type of the singularity $s \in S$. Then

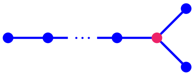
$$\mathcal{E}(\widetilde{M/G}; \tau, z) = \frac{1}{|G|}\mathcal{E}(M; \tau, z) + \sum_{s \in S} \lim_{\zeta \to 0} \mathcal{E}^{\mathrm{reg}}(\Gamma_s; \tau, z, \zeta).$$

Quotienting simple surface singularities

Recall from **[Hou/W]** the relations of the form

$$\mathcal{E}(D_{k+1};\tau,z,\zeta)=\tfrac{1}{2}\mathcal{E}(A_{2k-3};\tau,z,\zeta)+\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta).$$

In **[Hou/W]**, we give a geometric explanation:



D_{k+1}

– use **[Slodowy80]**:

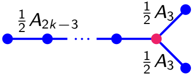
$\widetilde{\mathbb{C}^2/\Gamma}$ is the resolution of $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}/(\Gamma/\mathbb{Z}_2)$, where $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ is A_1;

Quotienting simple surface singularities

Recall from **[Hou/W]** the relations of the form

$$\mathcal{E}(D_{k+1};\tau,z,\zeta)=\tfrac{1}{2}\mathcal{E}(A_{2k-3};\tau,z,\zeta)+\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta).$$

In **[Hou/W]**, we give a geometric explanation:



D_{k+1}

– use **[Slodowy80]**:

$\widetilde{\mathbb{C}^2/\Gamma}$ is the resolution of $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}/(\Gamma/\mathbb{Z}_2)$,

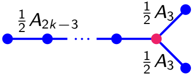
where $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ is A_1;

Quotienting simple surface singularities

Recall from **[Hou/W]** the relations of the form

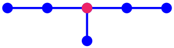
$$\mathcal{E}(D_{k+1};\tau,z,\zeta) = \tfrac{1}{2}\mathcal{E}(A_{2k-3};\tau,z,\zeta) + \mathcal{E}(A_3;\tau,z,\zeta) - \tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta).$$

In **[Hou/W]**, we give a geometric explanation:



D_{k+1}

– use **[Slodowy80]**:
$\widetilde{\mathbb{C}^2/\Gamma}$ is the resolution of $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}/(\Gamma/\mathbb{Z}_2)$,
where $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ is A_1;



E_6

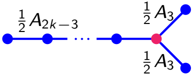
$$\mathcal{E}(E_6;\tau,z,\zeta) = \mathcal{E}(A_5;\tau,z,\zeta) + \tfrac{1}{2}\mathcal{E}(A_3;\tau,z,\zeta) - \tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta),$$

Quotienting simple surface singularities

Recall from **[Hou/W]** the relations of the form

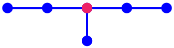
$$\mathcal{E}(D_{k+1};\tau,z,\zeta)=\tfrac{1}{2}\mathcal{E}(A_{2k-3};\tau,z,\zeta)+\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta).$$

In **[Hou/W]**, we give a geometric explanation:



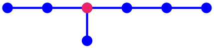
D_{k+1}

– use **[Slodowy80]**:
$\widetilde{\mathbb{C}^2/\Gamma}$ is the resolution of $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}/(\Gamma/\mathbb{Z}_2)$,
where $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ is A_1;



E_6

$$\mathcal{E}(E_6;\tau,z,\zeta)=\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta),$$



E_7

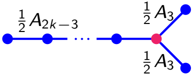
$$\mathcal{E}(E_7;\tau,z,\zeta)=\tfrac{1}{2}\mathcal{E}(A_7;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta),$$

Quotienting simple surface singularities

Recall from **[Hou/W]** the relations of the form

$$\mathcal{E}(D_{k+1};\tau,z,\zeta)=\tfrac{1}{2}\mathcal{E}(A_{2k-3};\tau,z,\zeta)+\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta).$$

In **[Hou/W]**, we give a geometric explanation:

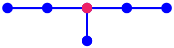


D_{k+1}

– use **[Slodowy80]**:

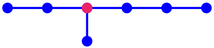
$\widetilde{\mathbb{C}^2/\Gamma}$ is the resolution of $\widetilde{\widetilde{\mathbb{C}^2/\mathbb{Z}_2}/(\Gamma/\mathbb{Z}_2)}$,

where $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ is A_1;



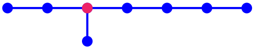
E_6

$$\mathcal{E}(E_6;\tau,z,\zeta)=\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta),$$



E_7

$$\mathcal{E}(E_7;\tau,z,\zeta)=\tfrac{1}{2}\mathcal{E}(A_7;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta),$$



E_8

$$\mathcal{E}(E_8;\tau,z,\zeta)=\tfrac{1}{2}\mathcal{E}(A_9;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_5;\tau,z,\zeta)+\tfrac{1}{2}\mathcal{E}(A_3;\tau,z,\zeta)-\tfrac{1}{2}\mathcal{E}(A_1;\tau,z,\zeta).$$

The End

Thank you
for your attention!

The End

Happy Birthday,
Uli!