Homotopy theory with stable ∞ -categories

Christoph Winges

LMU München/Universität Regensburg

From Analysis to Homotopy Theory 15th May 2024 Analysis: Topological K-theory and KK-theory

Kasparov:

- A, B C*-algebras $\rightsquigarrow KK(A, B)$ abelian group
- ► Kasparov product: $\mathsf{KK}(B, C) \otimes \mathsf{KK}(A, B) \to \mathsf{KK}(A, C)$
- $\blacktriangleright\,$ obtain category KK and functor $\mathrm{C}^*\mathrm{Alg}\to\mathsf{KK}$

$$\begin{split} & \mathcal{W}_{\mathsf{K}\mathsf{K}} := \text{morphisms in } \mathrm{C}^*\mathrm{Alg} \text{ which become isomorphisms in }\mathsf{K}\mathsf{K} \\ & \mathsf{Theorem } \left[\mathsf{Land-Nikolaus, } \mathsf{Uuye, } \mathsf{Higson, } \mathsf{Cuntz}\right] \\ & \mathsf{K}\mathsf{K}_\infty := \mathrm{C}^*\mathrm{Alg}[\mathcal{W}_{\mathsf{K}\mathsf{K}}^{-1}] \end{split}$$

- \blacktriangleright is a stable ∞ -category
- homotopy category identifies with KK,
- including triangulated structure due to Meyer-Nest

►
$$\mathrm{KU}(-) \simeq \hom_{\mathsf{KK}_{\infty}}(\mathbb{C}, -)$$

Homotopy Theory: Algebraic K-theory and localising motives

 $\mathrm{Cat}^{\mathrm{st}}:$ $\infty\text{-category}$ of small stable $\infty\text{-categories}$ and exact functors

Karoubi sequence: cofibre sequence

$$\mathfrak{C}\xrightarrow{i}\mathfrak{D}\xrightarrow{p}\mathfrak{E}$$

in Cat^{st} with *i* fully faithful

 ${\mathfrak X}$ stable $\infty\text{-category}$ with all small colimits, ${\it F}\colon {\rm Cat}^{\rm st}\to {\mathfrak X}$

- ▶ *localising*: *F* sends Karoubi sequences to cofibre sequences
- finitary: F preserves filtered colimits

Theorem [Blumberg–Gepner–Tabuada]

There exists a finitary localising invariant $\mathcal{U}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Mot}_{\mathrm{loc}}$ such that every finitary localising invariant factors uniquely over \mathcal{U} .

Algebraic K-theory and localising motives II

 $\mathrm{Cat}^{\mathrm{st}}$ has a symmetric monoidal structure \otimes with unit object Sp^ω

Blumberg–Gepner–Tabuada:

 ${\rm Mot}_{\rm loc}$ carries a symmetric monoidal structure such that ${\mathcal U}$ refines to a symmetric monoidal functor

Algebraic K-theory:

- $\mathsf{K}:=\hom_{\mathrm{Mot}_{\mathrm{loc}}}(\mathfrak{U}(\mathrm{Sp}^\omega),-)\colon\mathrm{Mot}_{\mathrm{loc}}\to\mathrm{Sp}$
 - Nat(ι, Ω[∞]F) ≃ Nat(K, F) for every finitary, localising
 F: Catst → Sp
 - K is canonically lax symmetric monoidal

Localising motives are a localisation

$$\label{eq:Wmot} \begin{split} & {\cal W}_{\rm mot} := {\sf morphisms \ in \ Cat}^{\rm st} \ {\sf which \ become \ equivalences \ in \ Mot}_{\rm loc} \\ & {\sf Theorem \ [Ramzi-Sosnilo-W]} \end{split}$$

$$\operatorname{Mot}_{\operatorname{loc}} \simeq \operatorname{Cat}^{\operatorname{st}}[W_{\operatorname{mot}}^{-1}]$$

Corollary

Every spectrum is the algebraic K-theory of a stable ∞ -category.

Proof

There exists a retraction diagram

$$\operatorname{Sp} \xrightarrow{\mathsf{tr}^*} \operatorname{Mod}_{\mathsf{K}(\mathbb{S})} \rightarrowtail \operatorname{Mot}_{\operatorname{loc}} \xrightarrow{\mathsf{K}} \operatorname{Sp}$$

 $\begin{array}{l} \blacktriangleright \ \mathcal{U}(\mathrm{Sp}^{\omega}) \text{ compact object, } \mathrm{End}(\mathcal{U}(\mathrm{Sp}^{\omega})) \simeq \mathsf{K}(\mathbb{S}) \\ \Rightarrow \mathsf{K}(\mathbb{S}) \text{-modules form a full subcategory of } \mathrm{Mot}_{\mathrm{loc}} \end{array}$

▶ tr: $K(S) \to S$ retraction to the unit $S \to K(S)$

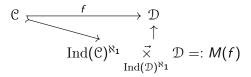
$\operatorname{Cat}^{\operatorname{st}}$ as a category of cofibrant objects

 $\operatorname{Cat}^{\operatorname{st}}$ becomes a category of cofibrant objects by choosing

- cofibrations = fully faithful exact functors
- weak equivalences = $W_{\rm mot}$

Key facts

- fully faithful functors are stable under pushout
- motivic equivalences are stable under pushout along cofibrations
- ► Tamme's lax pullback construction provides factorisations:



Identifying the localisation I

Cisinski:

- $\mathcal{M} := \operatorname{Cat}^{\operatorname{st}}[W_{\operatorname{mot}}^{-1}]$ has all small colimits
- ► localisation functor l: Catst → M preserves coproducts and pushouts along cofibrations

Hinich:

- \blacktriangleright $\mathcal M$ carries a symmetric monoidal structure
- \blacktriangleright localisation functor $\ell\colon {\rm Cat}^{\rm st}\to {\mathcal M}$ refines to a symmetric monoidal functor

$$\mathsf{Set}\ \mathrm{Calk}:= \textit{M}(\mathrm{Sp}^\omega \to 0)/\mathrm{Sp}^\omega. \quad \rightsquigarrow \quad \ell(\mathrm{Calk}) \otimes - \simeq \Sigma.$$

Identifying the localisation II

Kasprowski-W:

There exists a functor $\Gamma\colon \mathrm{Cat}^{\mathrm{st}}\to \mathrm{Cat}^{\mathrm{st}}$ such that

 $\mathcal{U}(\Gamma \mathcal{C}) \simeq \Omega \mathcal{U}(\mathcal{C}).$

$$\begin{array}{ll} \mathsf{Set}\ \mathsf{\Gamma} := \mathsf{\Gamma}(\mathrm{Sp}^{\omega}). \\ & \rightsquigarrow \quad \mathcal{U}(\mathsf{\Gamma} \otimes \mathfrak{C}) \simeq \mathcal{U}(\mathsf{\Gamma}) \otimes \mathcal{U}(\mathfrak{C}) \simeq \Omega \mathcal{U}(\mathfrak{C}) \\ & \rightsquigarrow \quad \mathcal{U}(\mathsf{\Gamma} \otimes \mathrm{Calk}) \simeq \mathcal{U}(\mathrm{Sp}^{\omega}) \ \mathsf{and} \ \mathcal{U}(\mathrm{Calk} \otimes \mathsf{\Gamma}) \simeq \mathcal{U}(\mathrm{Sp}^{\omega}) \end{array}$$

Blumberg–Gepner–Tabuada:

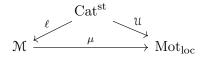
There exist motivic equivalences

$$\mathrm{Sp}^\omega \to \Gamma \otimes \mathrm{Calk} \quad \text{ and } \quad \mathrm{Sp}^\omega \to \mathrm{Calk} \otimes \Gamma$$

 $\Rightarrow \ell(\Gamma)\otimes -$ is an inverse to the suspension functor on ${\mathfrak M}$

Identifying the localisation III

Universal property induces



- ℓ inverts precisely $W_{\rm loc} \Rightarrow \mu$ conservative
- ▶ μ preserves small colimits $\Rightarrow \ell$ preserves filtered colimits

Upshot

 ℓ is a finitary localising invariant.

Size problems

Consider full subcategories of $\operatorname{Cat}^{\operatorname{st}}$ which are closed under all of the above operations.

The theorem of the heart

t-structures

- ► technical device to encode Postnikov towers in a stable ∞-category C
- Filtration quotients of Postnikov towers live in an abelian category, the *heart* C[♡]
- bounded t-structure: every object has a finite Postnikov tower
- \mathcal{A} abelian $\Rightarrow \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ has a bounded t-structure with $\mathcal{D}^{\mathrm{b}}(\mathcal{A})^{\heartsuit} \simeq \mathcal{A}$

Theorem of the heart [Barwick]

 ${\mathfrak C}$ stable $\infty\text{-category}$ with bounded t-structure, then

$$\tau_{\geq 0} \operatorname{\mathsf{K}}(\operatorname{\mathcal{D}^b}(\operatorname{\mathfrak{C}^{\heartsuit}})) \xrightarrow{\sim} \tau_{\geq 0} \operatorname{\mathsf{K}}(\operatorname{\mathfrak{C}})$$

The nonconnective case

Theorem [Antieau-Gepner-Heller]

 ${\ensuremath{\mathbb C}}$ stable $\infty\mbox{-category}$ with bounded t-structure and noetherian heart, then

$$\mathsf{K}(\mathcal{D}^{\mathrm{b}}(\mathfrak{C}^{\heartsuit})) \xrightarrow{\sim} \mathsf{K}(\mathfrak{C})$$

Theorem [Ramzi-Sosnilo-W]

There exists a stable $\infty\text{-category}$ with bounded t-structure ${\mathfrak C}$ such that

$$\mathsf{K}(\mathcal{D}^{\mathrm{b}}(\mathfrak{C}^{\heartsuit})) \to \mathsf{K}(\mathfrak{C})$$

is not an equivalence.

Sketch of proof

 ${\mathfrak C}$ stable $\infty\text{-category}\Rightarrow$ there exists a Karoubi sequence

$$\operatorname{Ac}(\operatorname{\mathfrak{C}}) \to \operatorname{\mathfrak{K}^b}(\operatorname{\mathfrak{C}}) \to \operatorname{\mathfrak{C}}$$

such that

- ► $\mathcal{K}^{b}(\mathcal{C})$ is the stabilisation of \mathcal{C} as an additive ∞-category $\Rightarrow \mathsf{K}(\mathcal{K}^{b}(\mathcal{C}))$ is $\mathsf{K}(\mathbb{Z})$ -local [Elmanto–Sosnilo]
- ► Ac(C) carries a bounded t-structure [Klemenc]

Upshot

theorem of the heart holds for $Ac(\mathcal{C})$ $\Rightarrow K(Ac(\mathcal{C})) \text{ is } K(\mathbb{Z})\text{-local}$ $\Rightarrow K(\mathcal{C}) \text{ is } K(\mathbb{Z})\text{-local}$

Theorem [Mitchell]

 $\mathsf{K}(\mathbb{Z})\otimes\mathsf{K}(n)\simeq 0$ for all primes p and all $n\geq 2$