Homotopy theory with stable ∞-categories

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From Analysis to Homotopy Theory
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Analysis: Topological K-theory and KK-theory

Kasparov:

- A, B C^*-algebras $\rightsquigarrow$ $\mathsf{KK}(A,B)$ abelian group
- Kasparov product: $\mathsf{KK}(B,C) \otimes \mathsf{KK}(A,B) \to \mathsf{KK}(A,C)$
- obtain category KK and functor $\mathrm{C^*Alg} \to \mathsf{KK}$

$W_{\mathsf{KK}} :=$ morphisms in $\mathrm{C^*Alg}$ which become isomorphisms in KK

Theorem [Land–Nikolaus, Uuye, Higson, Cuntz]

$\mathsf{KK}_\infty := \mathrm{C^*Alg}[W_{\mathsf{KK}}^{-1}]$

- is a stable ∞-category
- homotopy category identifies with KK,
- including triangulated structure due to Meyer–Nest
- $\mathrm{KU}(-) \simeq \hom_{\mathsf{KK}_\infty}(\mathbb{C}, -)$

Homotopy Theory: Algebraic K-theory and localising motives

$\mathrm{Cat}^{\mathrm{st}}$: ∞-category of small stable ∞-categories and exact functors

- *Karoubi sequence*: cofibre sequence

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

  in $\mathrm{Cat}^{\mathrm{st}}$ with i fully faithful

$\mathcal{X}$ stable ∞-category with all small colimits, $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{X}$

- *localising*: F sends Karoubi sequences to cofibre sequences
- *finitary*: F preserves filtered colimits

Theorem [Blumberg–Gepner–Tabuada]

There exists a finitary localising invariant $\mathcal{U}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Mot}_{\mathrm{loc}}$ such that every finitary localising invariant factors uniquely over $\mathcal{U}$.

Algebraic K-theory and localising motives II

$\mathrm{Cat}^{\mathrm{st}}$ has a symmetric monoidal structure $\otimes$ with unit object Sp^{ω}

Blumberg–Gepner–Tabuada:

$\mathrm{Mot}_{\mathrm{loc}}$ carries a symmetric monoidal structure such that $\mathcal{U}$ refines to a symmetric monoidal functor

Algebraic K-theory:

$\mathsf{K} := \hom_{\mathrm{Mot}_{\mathrm{loc}}}(\mathcal{U}(\mathrm{Sp}^{\omega}), -)\colon \mathrm{Mot}_{\mathrm{loc}} \to \mathrm{Sp}$

- $\mathrm{Nat}(\iota, \Omega^{\infty} F) \simeq \mathrm{Nat}(\mathsf{K}, F)$ for every finitary, localising $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$
- K is canonically lax symmetric monoidal

Localising motives are a localisation

$W_{\mathrm{mot}} :=$ morphisms in $\mathrm{Cat}^{\mathrm{st}}$ which become equivalences in $\mathrm{Mot}_{\mathrm{loc}}$

Theorem [Ramzi–Sosnilo–W]

$$\mathrm{Mot}_{\mathrm{loc}} \simeq \mathrm{Cat}^{\mathrm{st}}[W_{\mathrm{mot}}^{-1}]$$

Corollary

Every spectrum is the algebraic K-theory of a stable ∞-category.

Proof

There exists a retraction diagram

$$\mathrm{Sp} \xrightarrow{\mathsf{tr}^*} \mathrm{Mod}_{\mathsf{K}(\mathbb{S})} \rightarrowtail \mathrm{Mot}_{\mathrm{loc}} \xrightarrow{\mathsf{K}} \mathrm{Sp}$$

- $\mathcal{U}(\mathrm{Sp}^\omega)$ compact object, $\mathrm{End}(\mathcal{U}(\mathrm{Sp}^\omega)) \simeq \mathsf{K}(\mathbb{S})$
  $\Rightarrow$ $\mathsf{K}(\mathbb{S})$-modules form a full subcategory of $\mathrm{Mot}_{\mathrm{loc}}$
- $\mathsf{tr}\colon \mathsf{K}(\mathbb{S}) \to \mathbb{S}$ retraction to the unit $\mathbb{S} \to \mathsf{K}(\mathbb{S})$ $\square$

$\mathrm{Cat}^{\mathrm{st}}$ as a category of cofibrant objects

$\mathrm{Cat}^{\mathrm{st}}$ becomes a category of cofibrant objects by choosing

- cofibrations = fully faithful exact functors
- weak equivalences = W_{mot}

Key facts

- fully faithful functors are stable under pushout
- motivic equivalences are stable under pushout along cofibrations
- Tamme's lax pullback construction provides factorisations:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad f \quad} & \mathcal{D} \\ & \searrow & \uparrow \\ & & \mathrm{Ind}(\mathcal{C})^{\aleph_1} \underset{\mathrm{Ind}(\mathcal{D})^{\aleph_1}}{\vec{\times}} \mathcal{D} =: M(f) \end{array}$$

Identifying the localisation I

Cisinski:

- $\mathcal{M} := \mathrm{Cat}^{\mathrm{st}}[W_{\mathrm{mot}}^{-1}]$ has all small colimits
- localisation functor $\ell\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{M}$ preserves coproducts and pushouts along cofibrations

Hinich:

- $\mathcal{M}$ carries a symmetric monoidal structure
- localisation functor $\ell\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{M}$ refines to a symmetric monoidal functor

Set $\mathrm{Calk} := M(\mathrm{Sp}^{\omega} \to 0)/\mathrm{Sp}^{\omega}$. $\quad\rightsquigarrow\quad \ell(\mathrm{Calk}) \otimes - \simeq \Sigma$.

Identifying the localisation II

Kasprowski–W:

There exists a functor $\Gamma\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{st}}$ such that

$$\mathcal{U}(\Gamma\mathcal{C}) \simeq \Omega\mathcal{U}(\mathcal{C}).$$

Set $\Gamma := \Gamma(\mathrm{Sp}^{\omega})$.

$\rightsquigarrow \quad \mathcal{U}(\Gamma \otimes \mathcal{C}) \simeq \mathcal{U}(\Gamma) \otimes \mathcal{U}(\mathcal{C}) \simeq \Omega\mathcal{U}(\mathcal{C})$

$\rightsquigarrow \quad \mathcal{U}(\Gamma \otimes \mathrm{Calk}) \simeq \mathcal{U}(\mathrm{Sp}^{\omega})$ and $\mathcal{U}(\mathrm{Calk} \otimes \Gamma) \simeq \mathcal{U}(\mathrm{Sp}^{\omega})$

Blumberg–Gepner–Tabuada:

There exist motivic equivalences

$$\mathrm{Sp}^{\omega} \to \Gamma \otimes \mathrm{Calk} \quad \text{and} \quad \mathrm{Sp}^{\omega} \to \mathrm{Calk} \otimes \Gamma$$

$\Rightarrow \ell(\Gamma) \otimes -$ is an inverse to the suspension functor on $\mathcal{M}$

Identifying the localisation III

Universal property induces

$$\begin{array}{ccc} & \mathrm{Cat}^{\mathrm{st}} & \\ {}^{\ell}\swarrow & & \searrow^{\mathcal{U}} \\ \mathcal{M} & \xrightarrow{\quad\mu\quad} & \mathrm{Mot}_{\mathrm{loc}} \end{array}$$

- ℓ inverts precisely $W_{\mathrm{loc}} \Rightarrow \mu$ conservative
- μ preserves small colimits $\Rightarrow \ell$ preserves filtered colimits

Upshot

ℓ is a finitary localising invariant.

Size problems

Consider full subcategories of $\mathrm{Cat}^{\mathrm{st}}$ which are closed under all of the above operations.

The theorem of the heart

t-structures

- technical device to encode Postnikov towers in a stable ∞-category $\mathcal{C}$
- filtration quotients of Postnikov towers live in an abelian category, the *heart* $\mathcal{C}^\heartsuit$
- *bounded* t-structure: every object has a finite Postnikov tower
- $\mathcal{A}$ abelian $\Rightarrow$ $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ has a bounded t-structure with
$$\mathcal{D}^{\mathrm{b}}(\mathcal{A})^\heartsuit \simeq \mathcal{A}$$

Theorem of the heart [Barwick]

$\mathcal{C}$ stable ∞-category with bounded t-structure, then

$$\tau_{\geq 0}\,\mathsf{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C}^\heartsuit)) \xrightarrow{\sim} \tau_{\geq 0}\,\mathsf{K}(\mathcal{C})$$

The nonconnective case

Theorem [Antieau–Gepner–Heller]

$\mathcal{C}$ stable ∞-category with bounded t-structure and noetherian heart, then

$$\mathrm{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C}^{\heartsuit})) \xrightarrow{\sim} \mathrm{K}(\mathcal{C})$$

Theorem [Ramzi–Sosnilo–W]

There exists a stable ∞-category with bounded t-structure $\mathcal{C}$ such that

$$\mathrm{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C}^{\heartsuit})) \to \mathrm{K}(\mathcal{C})$$

is not an equivalence.

Sketch of proof

$\mathcal{C}$ stable ∞-category $\Rightarrow$ there exists a Karoubi sequence

$$\mathrm{Ac}(\mathcal{C}) \to \mathcal{K}^{\mathrm{b}}(\mathcal{C}) \to \mathcal{C}$$

such that

- $\mathcal{K}^{\mathrm{b}}(\mathcal{C})$ is the stabilisation of $\mathcal{C}$ as an additive ∞-category
  $\Rightarrow$ $\mathrm{K}(\mathcal{K}^{\mathrm{b}}(\mathcal{C}))$ is $\mathrm{K}(\mathbb{Z})$-local [Elmanto–Sosnilo]
- $\mathrm{Ac}(\mathcal{C})$ carries a bounded t-structure [Klemenc]

Upshot

theorem of the heart holds for $\mathrm{Ac}(\mathcal{C})$
$\Rightarrow$ $\mathrm{K}(\mathrm{Ac}(\mathcal{C}))$ is $\mathrm{K}(\mathbb{Z})$-local
$\Rightarrow$ $\mathrm{K}(\mathcal{C})$ is $\mathrm{K}(\mathbb{Z})$-local

Theorem [Mitchell]

$\mathrm{K}(\mathbb{Z}) \otimes K(n) \simeq 0$ for all primes p and all $n \geq 2$